New Horizons in Dispersive Hydrodynamics

Isaac Newton Institute for Mathematical Sciences

Dispersive Shock Waves in atmospheric and oceanic events. An accurate description of these phenomena through Whitham-Boussinesq water wave models.

Joint work with N. Smyth[1] and T. Marchant[2]

Noel F. Smyth

Tim Marchant

Dra. Rosa Maria Vargas Magaña

https://rosavargas.github.io/

School of Mathematics, University of Edinburgh

Postdoctoral Researcher, CONACyT

THE UNIVERSITY OF EDINBURGH

[1] **School of Mathematics, University of Edinburgh** [2] **Australian Mathematical Sciences Institute, University of Melbourne**

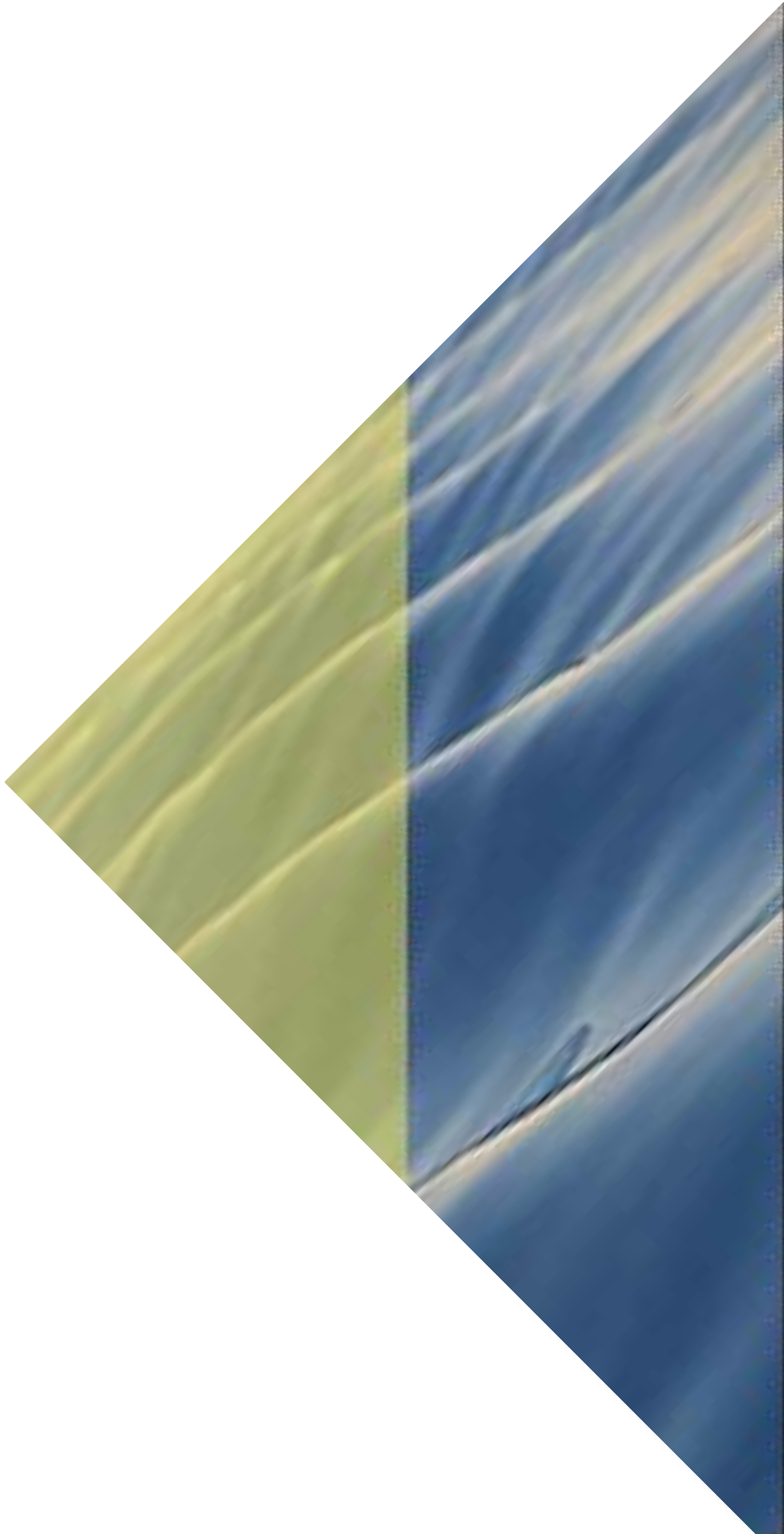
Outline

► Geophysical phenomena that involves DSW

► Anatomy of a DSW of KdV-type and the Dispersive Shock Fitting Method applied to four weakly nonlinear water waves models for the description of the macroscopic properties on these Bores.

► Fully nonlinear water waves - *undular bores*

► **Results** (The analytical and numerical findings to stablish some results on the applicability for describing oceanic events and on the agreement of the DS-fitting method relative to the fully non-linear water wave models.)



Geophysical phenomena that involves **DSW**.

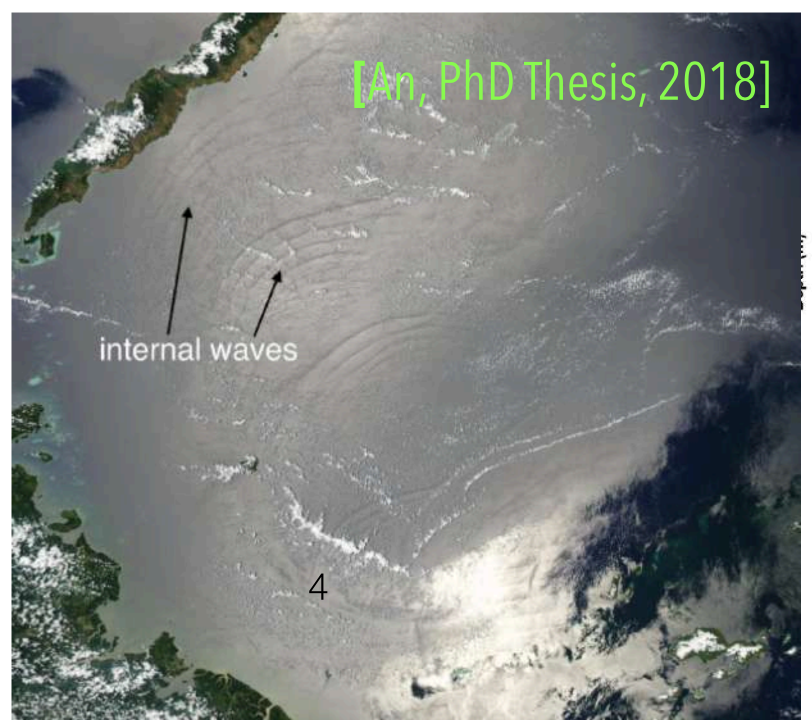
Dispersive Shock Waves are everywhere!

[El-Hoefer, 2016]

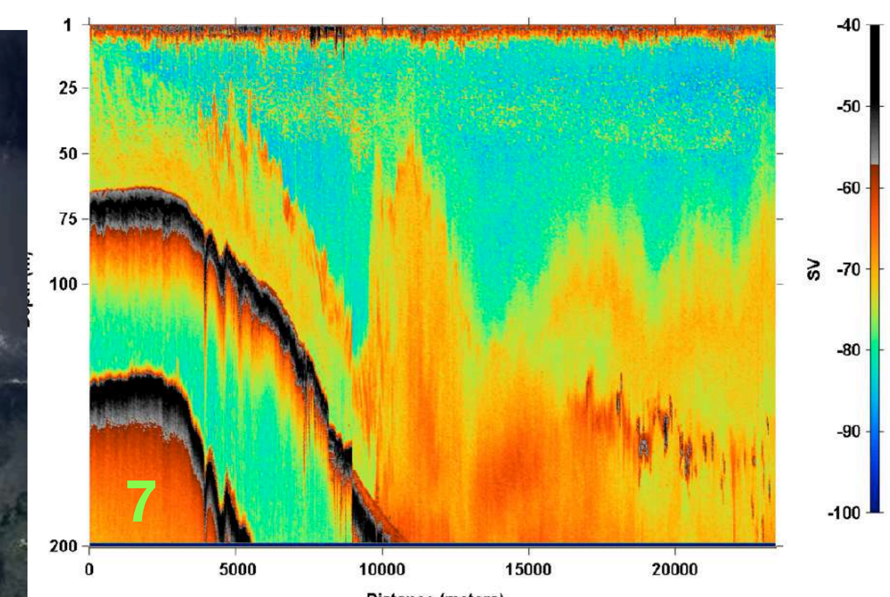
[An, PhD Thesis, 2018]

1. Morning glory roll cloud. **2. Atmospheric gravity wave**s moving southward off the Texas coast and out over the western Golf of Mexico **3.** Three standing paddle board riders on a river **undular bore** in Turnagain Arm, Alaska. **4. Tsunami-like waves hitting North Beach in Koh Pu Thailand** 2004 **6. Internal Solitary Waves.** Even the eye at sea level can detect the induced surface-wave changes as rough and smooth regions. In the Sulu sea between Philippines and Malaysia. **7. Lee wave** from the northeast peak of Georges Bank out into Georges Basin **seen** with echo sounder de 200 kHz.

[Copyright Anders Grawin, 2004]



[An, PhD Thesis, 2018]

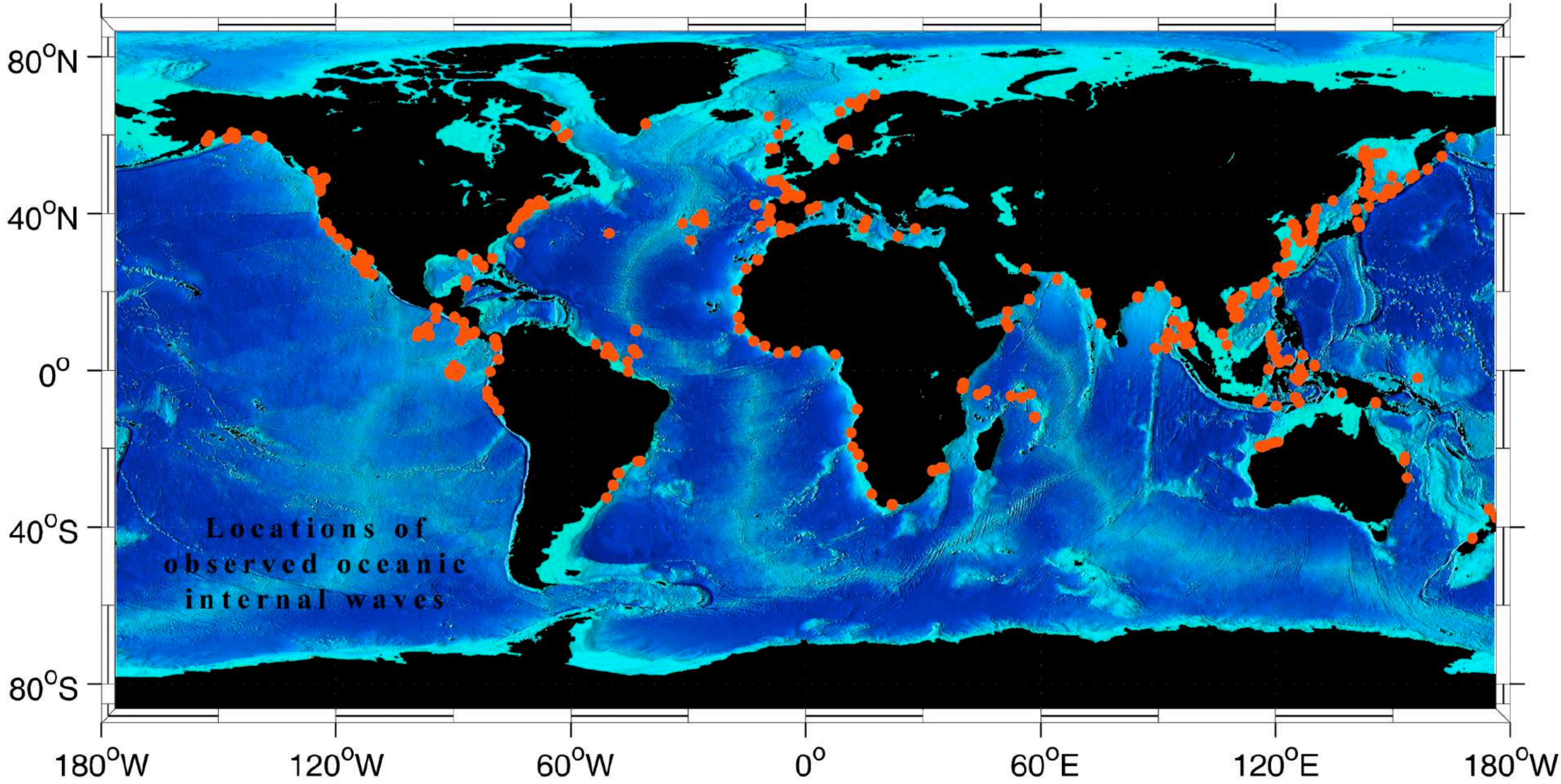


[Wiebe- Irish- Beardsley-Stanton,2000]

Dispersive Shock Waves play an important role in oceanic events!

- coastal tsunami propagation
- Internal ocean transport
- Equilibrium of costal ecosystems

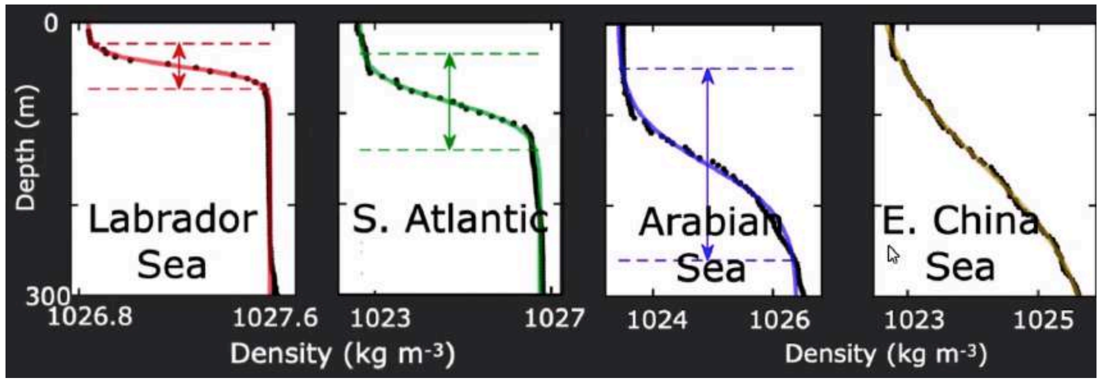
The vast majority are satellite images!



An Atlas of Oceanic Internal Waves and Solitons (May 2002) by Global Ocean AssociatesPrepared for Office of Naval Research – Code 322 PO5F Pulished in : _doi=10.1.1.737.631&rep=rep1&type=pdf

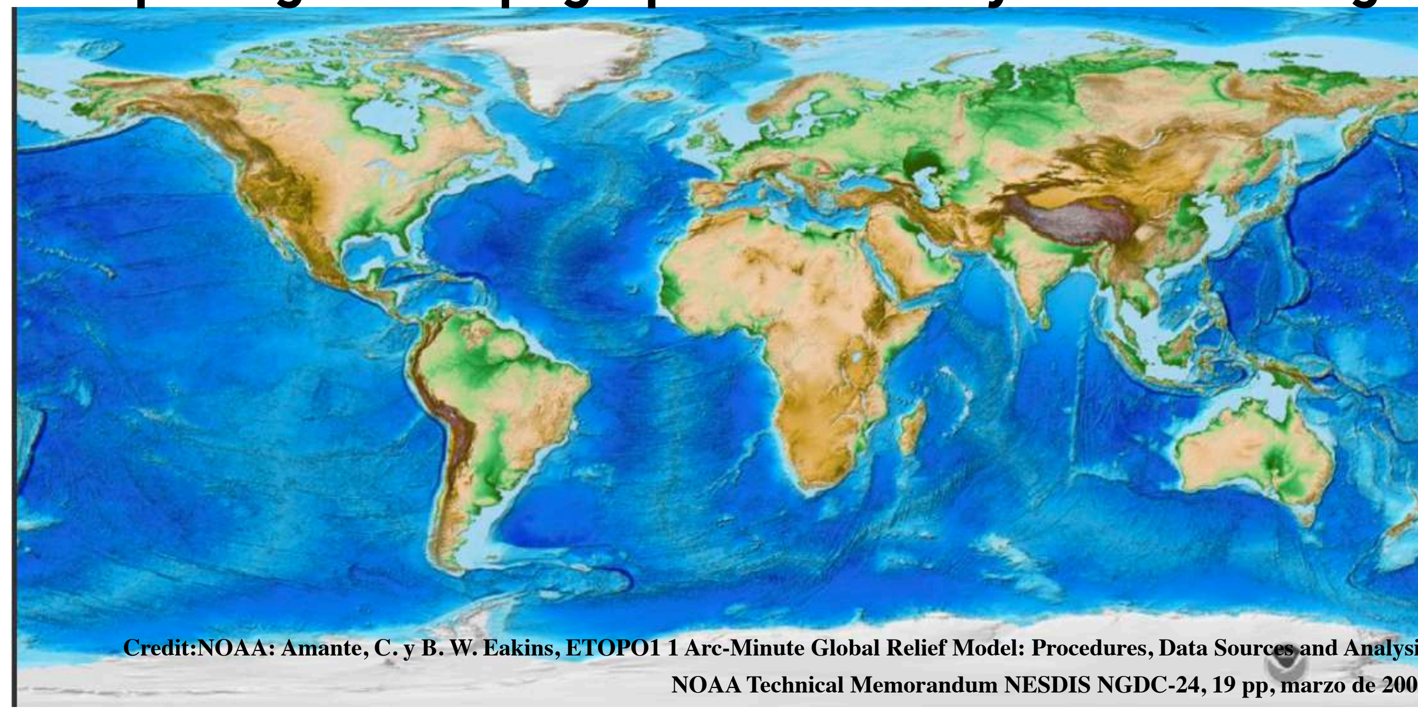
Complex Physical settings **that can perturb DSW**

1. Different Stratification in the ocean

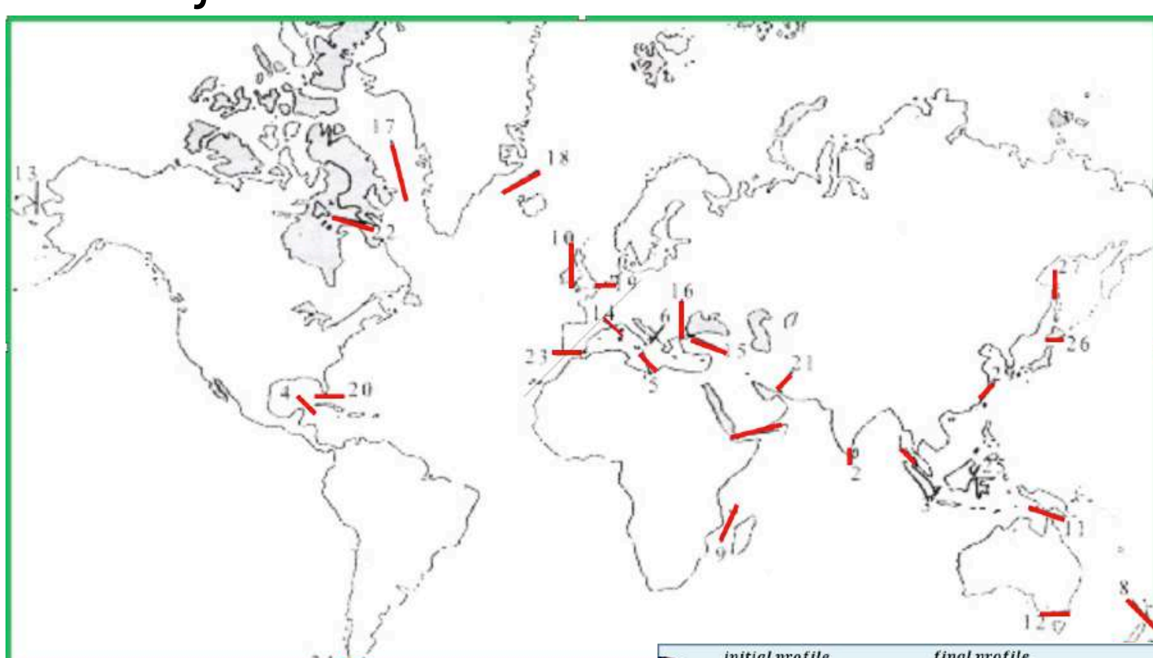


[Vieira-Allshouse, 2020]Credit :Magda Carr Talk

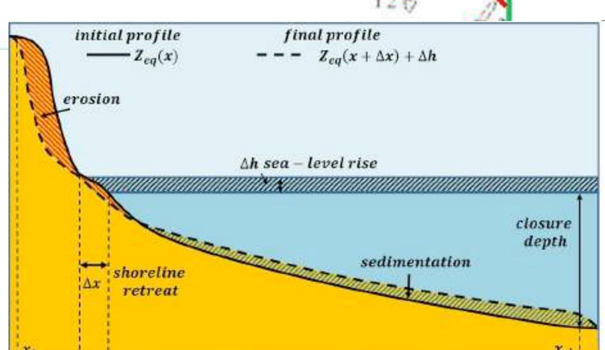
Complete global topographic and bathymetric coverage



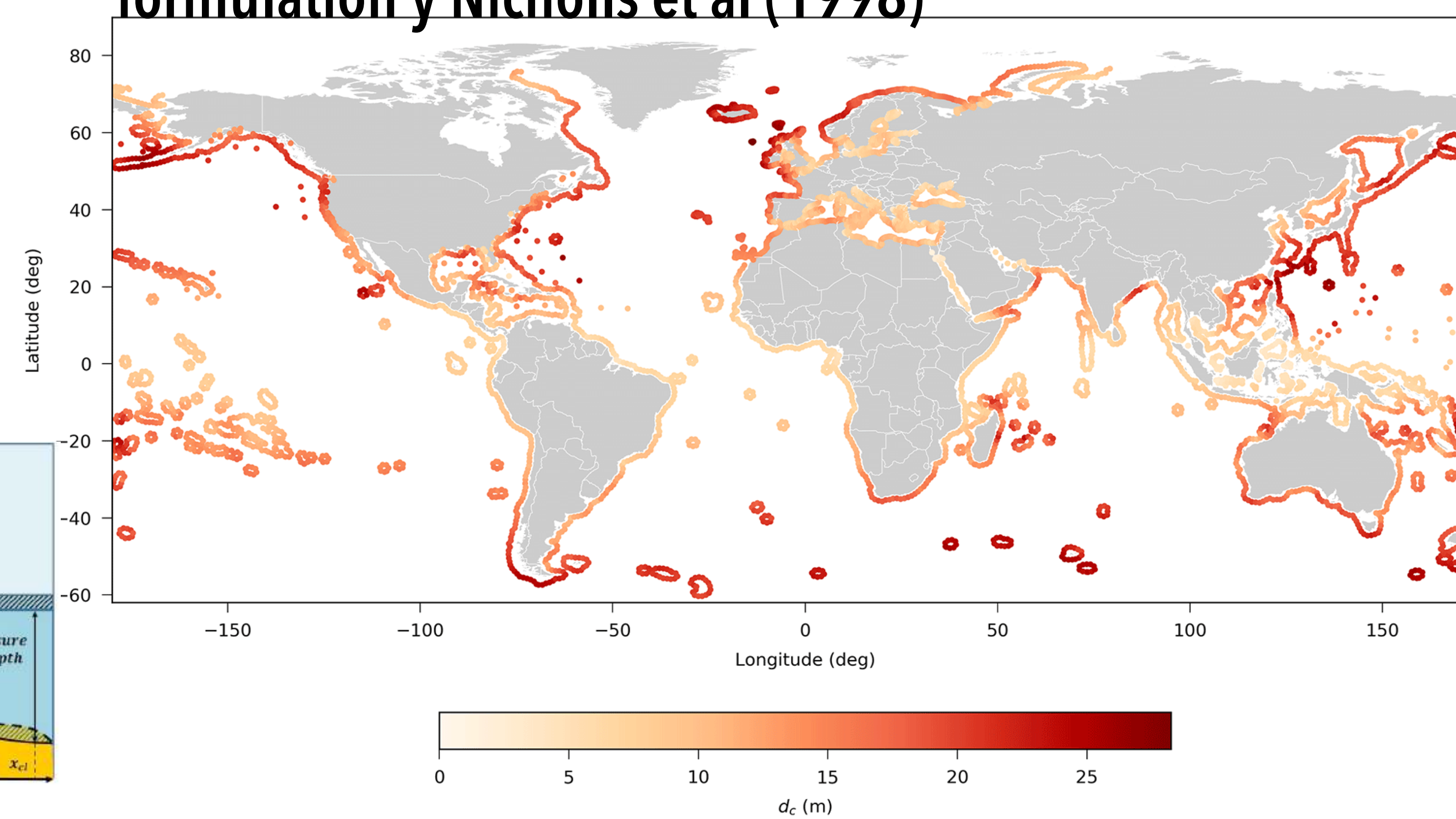
26 Major Straits in the world



https://www.jagranjosh.com/general-knowledge/list-of-the-major-straits-of-the-world-1495794208-1



Depths of closure dc along the global coastline using formulation y Nicholls et al (1998)



¿Por qué es relevante la obtención de estos modelos simplificados?

- 🌊 Gran desarrollo tecnológico en la instrumentación de medición **in situ** y en **laboratorio** de estos fenómenos.

- 🌊 Enorme capacidad de cómputo y el diseño de esquemas numéricos que preservan propiedades estructurales de las ecuaciones que ha permitido integrar numéricamente las ecuaciones por tiempos prolongados.

- 🌊 Se han registrado y documentado los **efectos no lineales** de la combinación de la dispersión de las ondas y la topografía y las fuerzas de restauración que se manifiestan ante la **propagación de ondas por** periodos prolongados del tiempo.

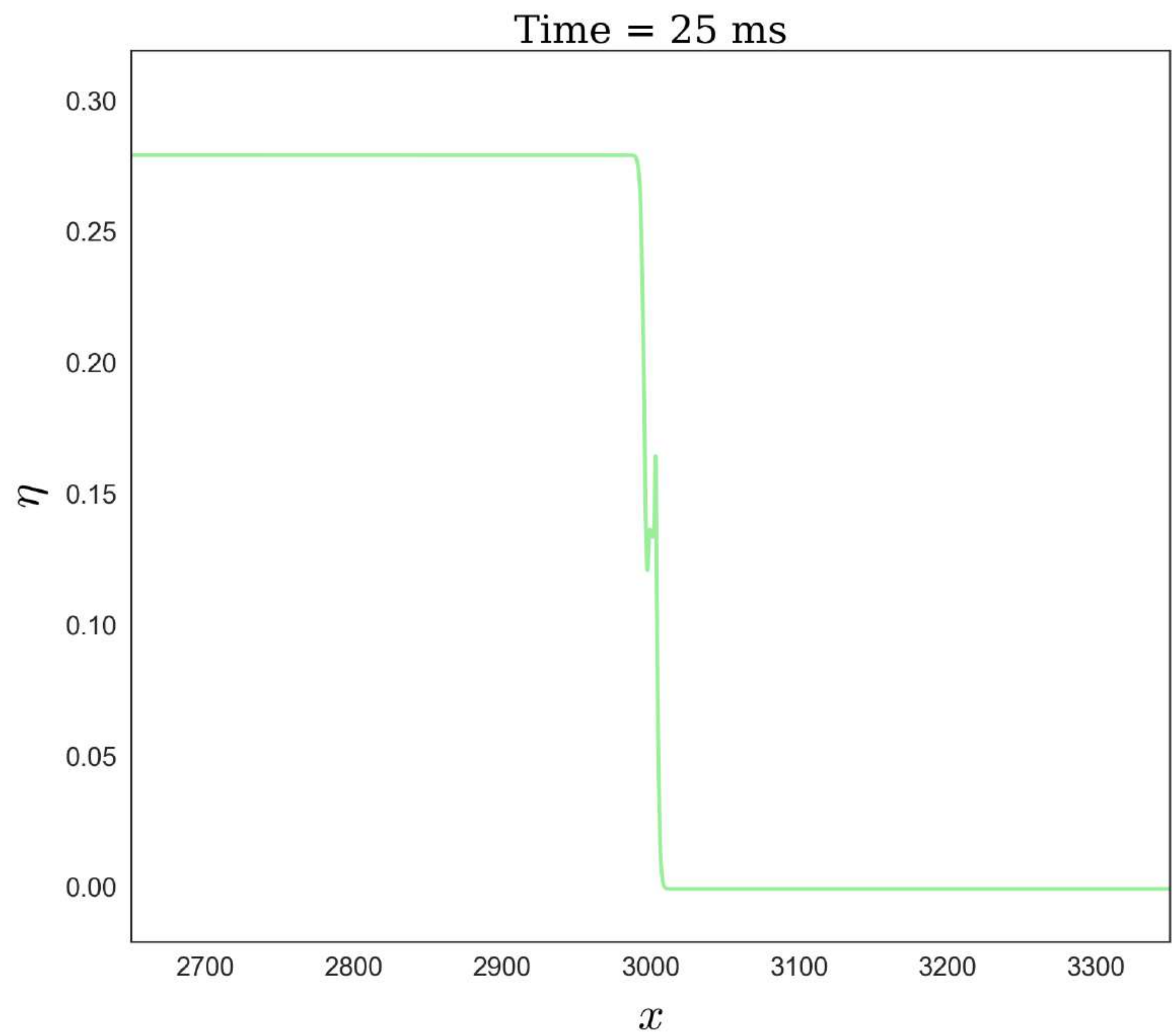
- 🌊 Se han observado numéricamente **ondas coherentes estables e inestables** y los **efectos de radiación, resonancia, disipación, vorticidad, enfoque y desenfoque** de las ondas, tren de ondas periódicos, frentes de ondas etc.

► DSW are of Interdisciplinary interest

► Simpler models that captures with loyalty relevant physical situations can lead us to the discovery of structural properties, and in particular for DSW can give us some insights one the long time evolution of these waves under complex physical settings.

DSW are :

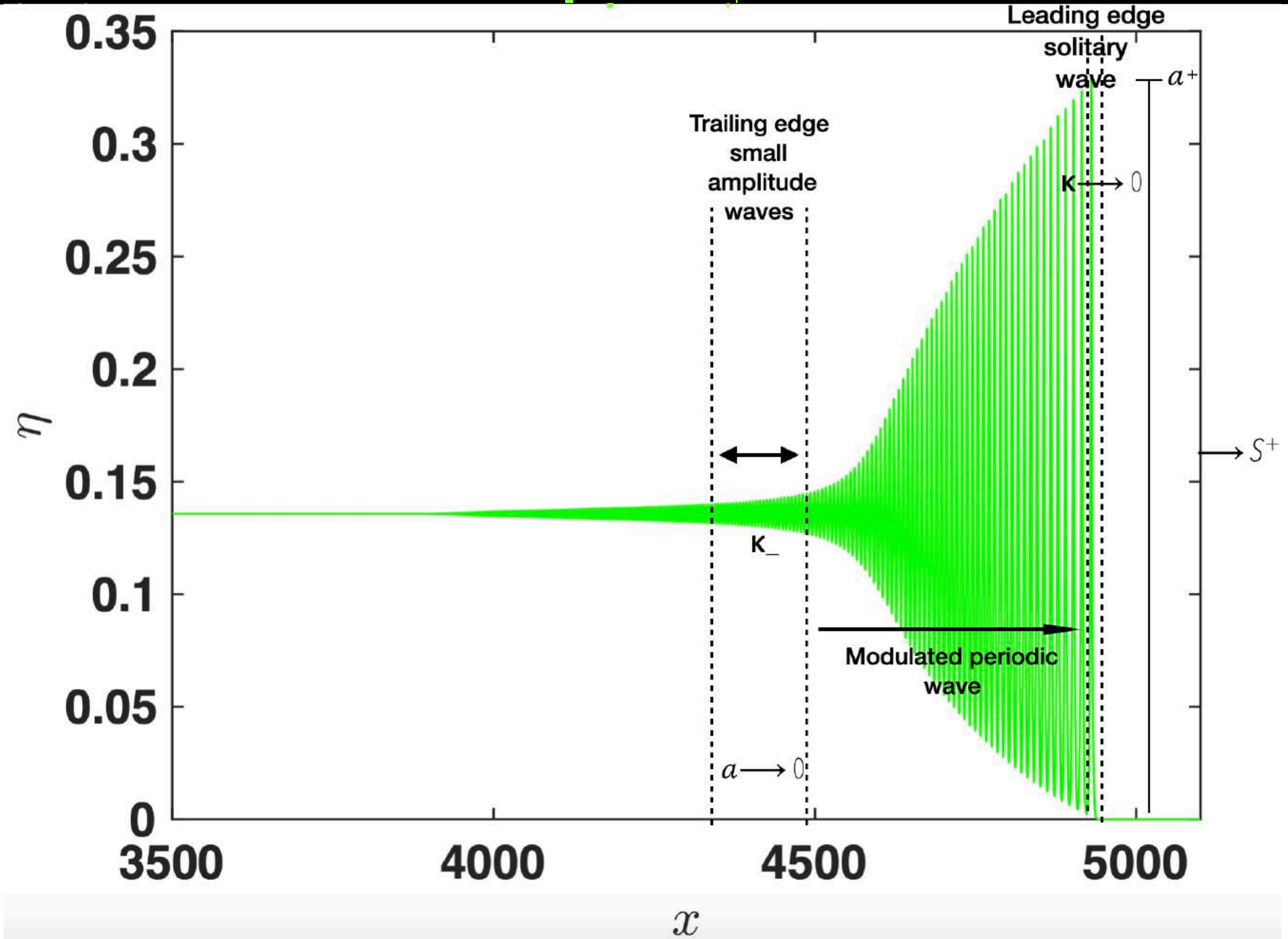
- A generic type of wave phenomenon arising as solutions of nonlinear dispersive wave equations.
- They are of multiscale nature.
- They are transient solutions.



DSW is an expanding, slowly-modulated wave form, with a linear wave-train at one end and a solitary wave at the other.

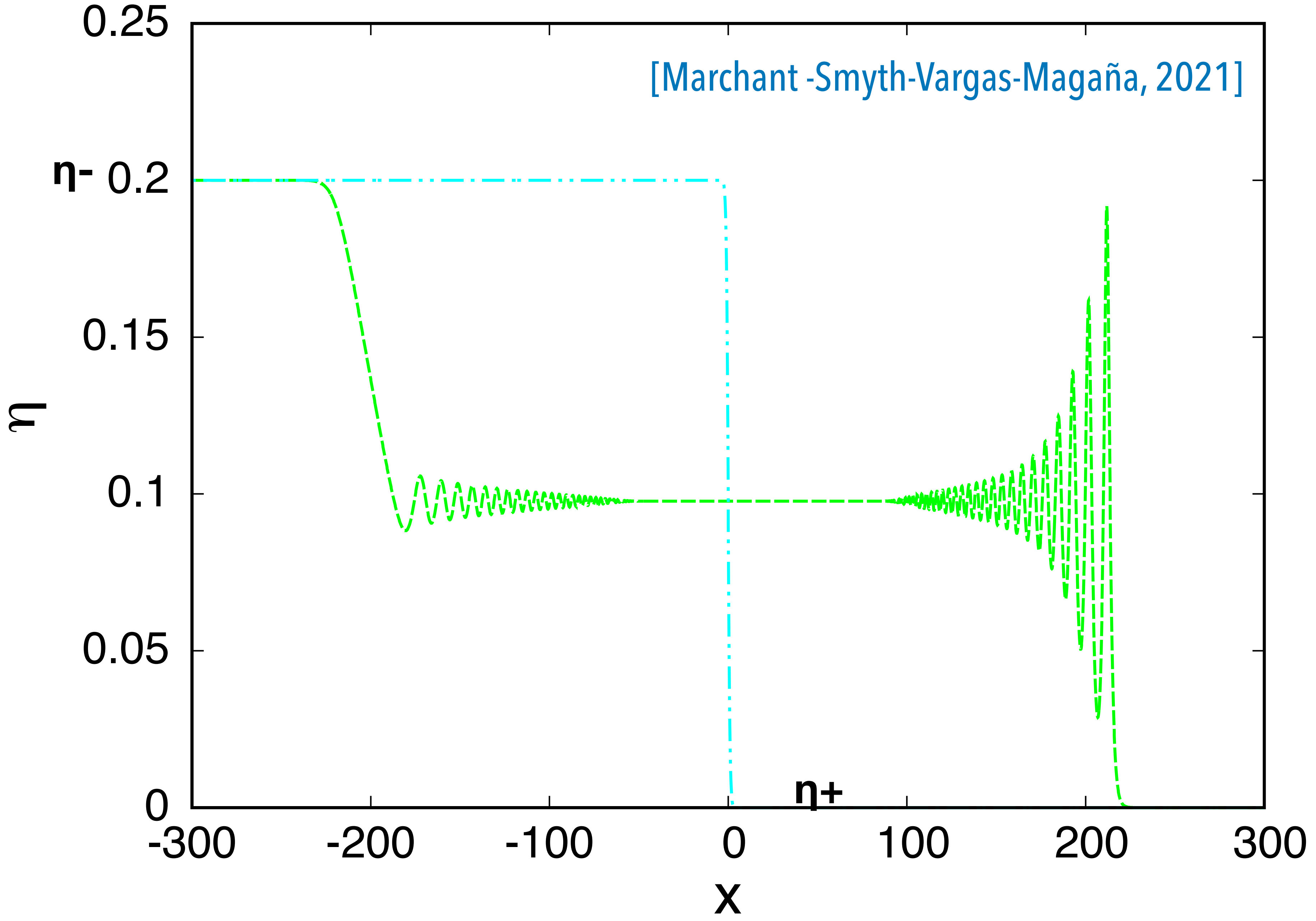
Anatomy of a DSW of KdV-type

Macroscopic Properties of a DSW



It consists of two edges the trailing edge and leading edge **propagating with different speeds with a modulated dispersive wavetrain between these**. The two distinct speeds of propagation correspond to two limits, the harmonic (linear) wave limit where the wave amplitude is small and the solitary wave limit where the wavenumber is small

Anatomy of a DSW



This is a typical undular bore solutions of these four Boussinesq systems. This corresponds with the Whitham-Boussinesq-Hamiltonian, for the initial condition (cyan dashed line).

Current Methods to determine analytically the **macroscopic DSW properties.**

- **Nonlinear wave modulation theory:** To obtain the full undular bore solution of a nonlinear, dispersive wave equation, the Whitham modulation equations for this equation need to be known.

- **Dispersive Shock Fitting Method** introduced by G. El and collaborators in 2005

For **Dispersive Shock-Fitting** method to be applied we need to check:

1- The **nonlinear, dispersive equations must satisfy** the following conditions**:**

(i) It supports **periodic travelling-wave solutions,** parameterized by three independent variables.

(ii) It possesses **at least two conservation laws**.

(iii) It admits a **dispersionless (hydraulic) limit** obtained by introducing the slow variables.

(amplitude, wave number, phase wave)

(iv) The linear dispersion relation **ω(k) is real-valued.**

And

Representation of the modulation system , obtained by averaging two conservation laws and wave conservation $k_t + \omega_x = 0$

$$\mathbf{U}_t + A(\mathbf{U})\mathbf{U}_x = 0$$

$$\mathbf{U}(\mathbf{x}, t) = (\bar{u}, k, \tilde{\kappa})^T$$

$A(\mathbf{U})$ Is the Jacobian Matrix

1- The **Whitham system must satisfy** the following conditions:

(v) Is strictly hyperbolic.

Hyperbolicity cannot be checked without directly solving the Whitham system so this condition should be verified by, for example, comparison of theoretical results with numerical simulations.

(v) The Whitham system is genuinely nonlinear.

Degenerate form of the Whitham modulation system

$$\mathbf{U}_t + A(\mathbf{U})\mathbf{U}_x = 0$$

In the harmonic limit $\tilde{k} = 0$

In the soliton limit $k = 0$

$$\begin{aligned} \tilde{k} &= 0 \\ \bar{u}_t + V(\bar{u})\bar{u}_x &= 0 \\ \kappa_t + \omega_0(\kappa, \bar{u})_x &= 0 \end{aligned}$$

$$\Lambda = \frac{k}{\tilde{k}}$$

$$\frac{\partial \Lambda}{\partial t} + \frac{\tilde{\omega}_s}{\tilde{k}}\frac{\partial \Lambda}{\partial x} + \frac{\Lambda}{\tilde{k}}\frac{\partial \bar{u}}{\partial x}\left\{\frac{d\tilde{k}}{d\bar{u}}\left(\frac{\partial \tilde{\omega}_s}{\partial \tilde{k}} - \bar{u}\right) + \frac{\partial \tilde{\omega}_s}{\partial \bar{u}}\right\} = \mathcal{O}\left(\Lambda \frac{\partial \Lambda}{\partial x}\right)$$

Matching Ordinary Differential equations

$$\mathbf{U}_t + A(\mathbf{U})\mathbf{U}_x = 0$$

The equation for the wavenumber k=k(u)
at the trailing edge of the DSW is

In the harmonic limit $\tilde{k} = 0$

$$\frac{dk}{d\bar{u}} = \frac{\partial\omega_0/\partial\bar{u}}{V(\bar{u}) - \partial\omega_0/\partial k}$$

$$k(u_s) = 0$$

In the soliton limit $k = 0$

$$\frac{d\tilde{k}}{d\bar{u}} = \frac{\partial\tilde{\omega}_0/\partial\bar{u}}{V(\bar{u}) - \partial\tilde{\omega}_0/\partial k}$$

$$\tilde{k}(u_h) = 0$$

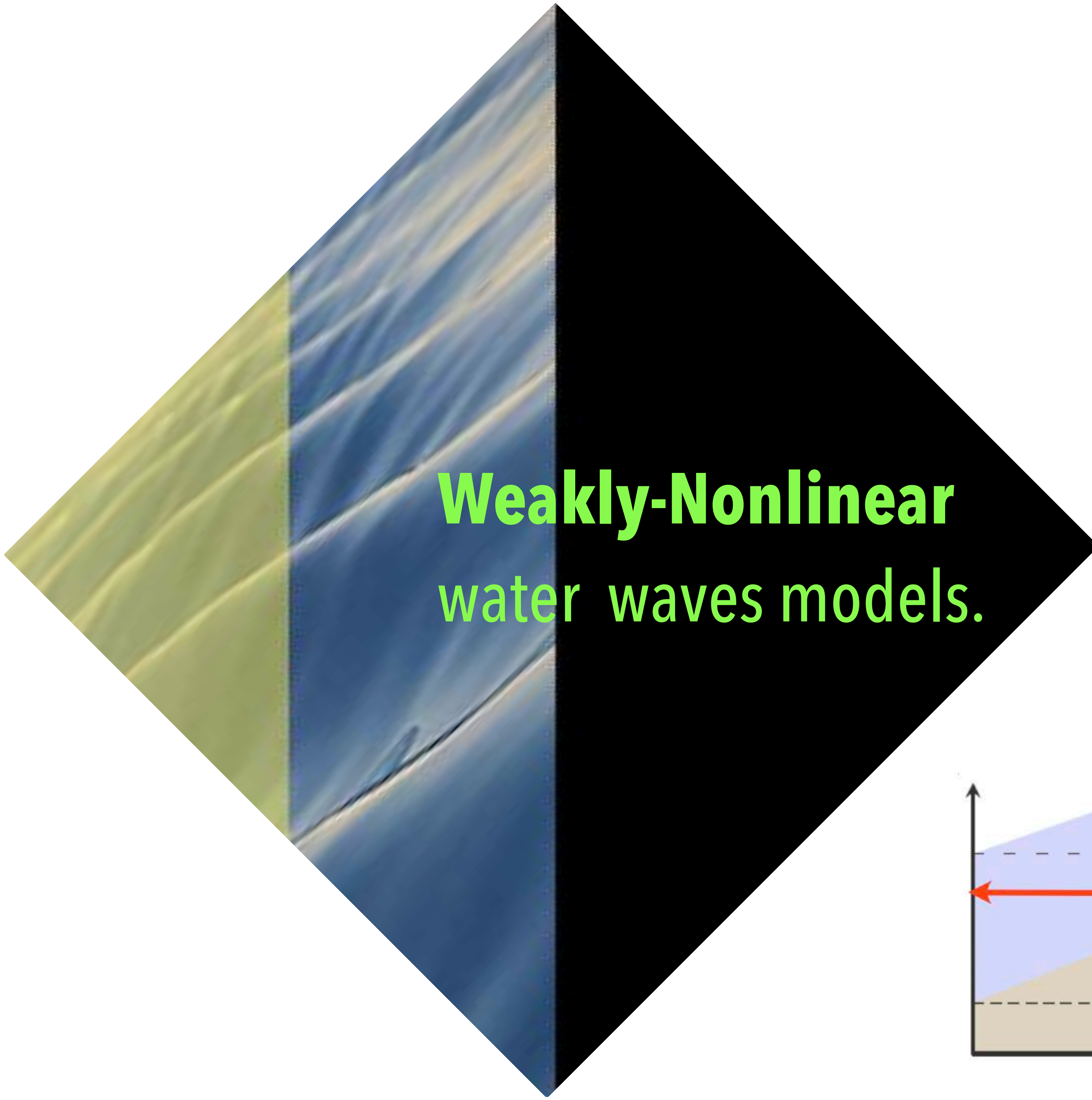
In this work, using DS-**Fitting Method** we are **reporting:**

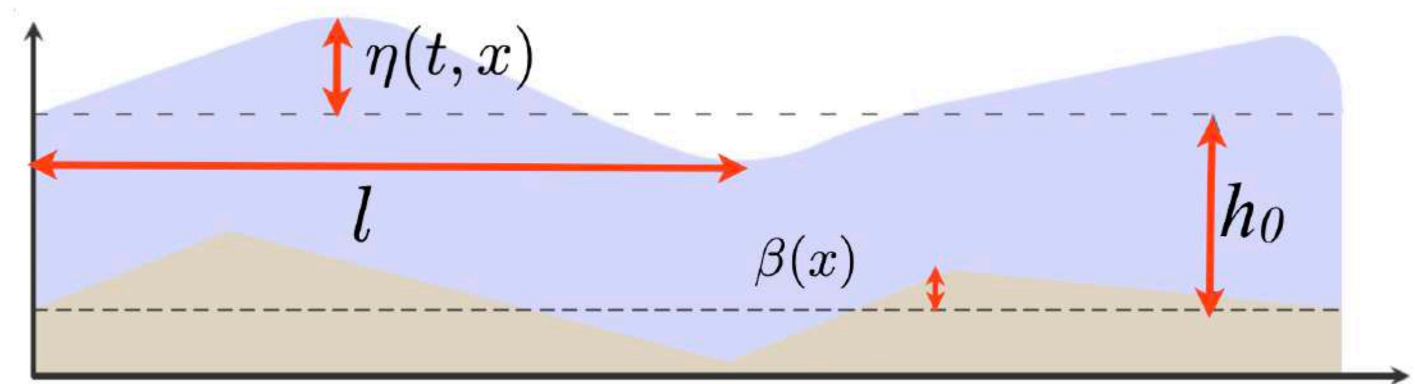
- Leading edge solitary amplitude
- Leading edge solitary velocity
- Trailing edge harmonic wave number
- Intermediate amplitude.
  Threshold of "Benjamin-Feir instability"

For the four Boussinesq systems studied in this work:

- **Sistema A:** **Standard Boussinesq system**
- **Sistema B:** **Boussinesq-Hamiltonian system**
- **Sistema C:** **Full-dispersion shallow water equations**
- **Sistema D:** **Whitham-Boussinesq System**

Weakly-Nonlinear water waves models.





$$\delta = \frac{h_0^2}{l^2} \ll 1 \quad \sim \quad \epsilon = \frac{\eta}{h_0}$$ **small!**

History bites on the Water Waves Theory

Hamiltonian Formulation of the Water waves problem

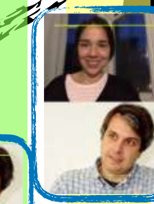
1993

C. Sulem
W. Craig

1968

Zakharov

2016



2013

1967

Whitham Equation

1895

KdV-equation

1872

Boussinesq-Equations

Explicit Introduction of the Dirichlet-Neumann operator in the Hamiltonian.
Craig-Groves: Rigourous derivation of the Classical long-wave models

Whitham-Boussinesq model
Finite Constant depth
[Aceves-Panayotaros,2014]

Whitham-Boussinesq model
Variable-depth
[VargasMagaña-Panayotaros,2016]

1755

Euler Equations

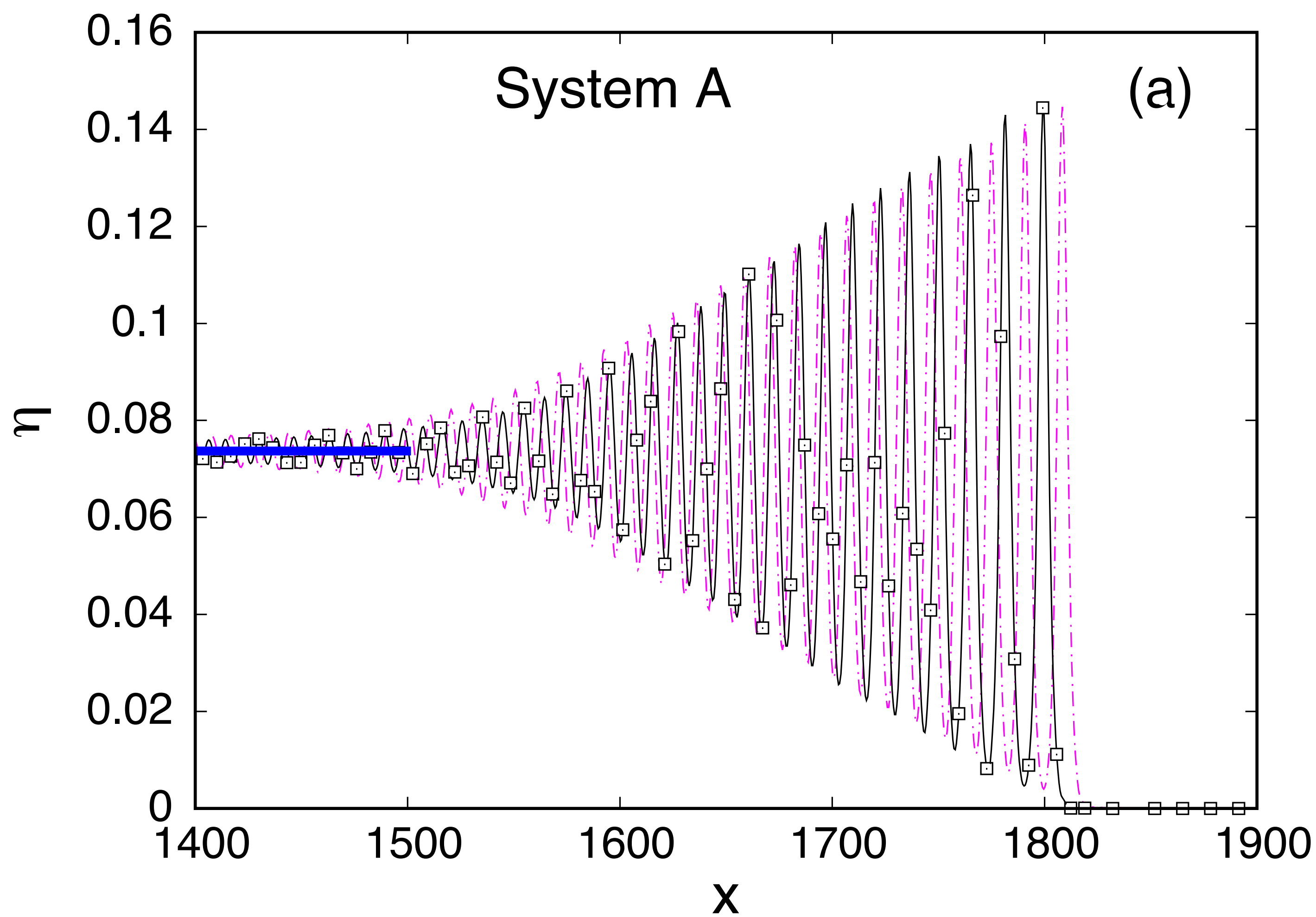
Remarkable properties of Whitham-Boussinesq model

- Same dispersion relation of Surface Euler Equations
- Existence of Traveling waves
- Breaking phenomena
- Existence of Solitary waves [Nilsoon-Wang, 2019],
- Existence of Stokes waves [Vargas-PhD Thesis 2017]
- Peaking phenomena (a logarithmically cusped of greatest height) [2019]
- **Better description of short-waves** [Smyth-Marchant-VargasMagaña,2021]
- Hamiltonian system (this is suitable to control long time numerics) [Aceves-Panayotaros, **2014**] [Vargas-Panayotaros, 2016]
- Numerically tractable with numerical schemes that preserves mathematical structure [Sulem-Craig 1993], [Guyenne-Nicholls 2005] [Guyenne, 2017] [Vargas-Panayotaros,2016]
- Extendable to 3D-waves [Andrade-Nachbin 2019]
- Extendable to high order variable depth [Vargas-Panayotaros,2016]
- Extendable to surface tension effects [Kalish-Pilod,2019]
- Local well-posedness [Pei-Wang, 2019], [Klein-Linares-Pilod-Saut, 2018]
- Validity for modelqinng waves on shallow water [Carter, 2018]
- Numerical bifurcation and spectral stability [Claassen-Johnson,2019]

Sistema A: Standard Boussinesq system

$$\eta_t = -u_x - (\eta u)_x,$$

$$u_t = -uu_x - \eta_x - \frac{1}{3}\eta_{xxx}$$



*Thik Line blueThe position of the trailing edge of the undular bore

$$V = \sqrt{1 + a}.$$

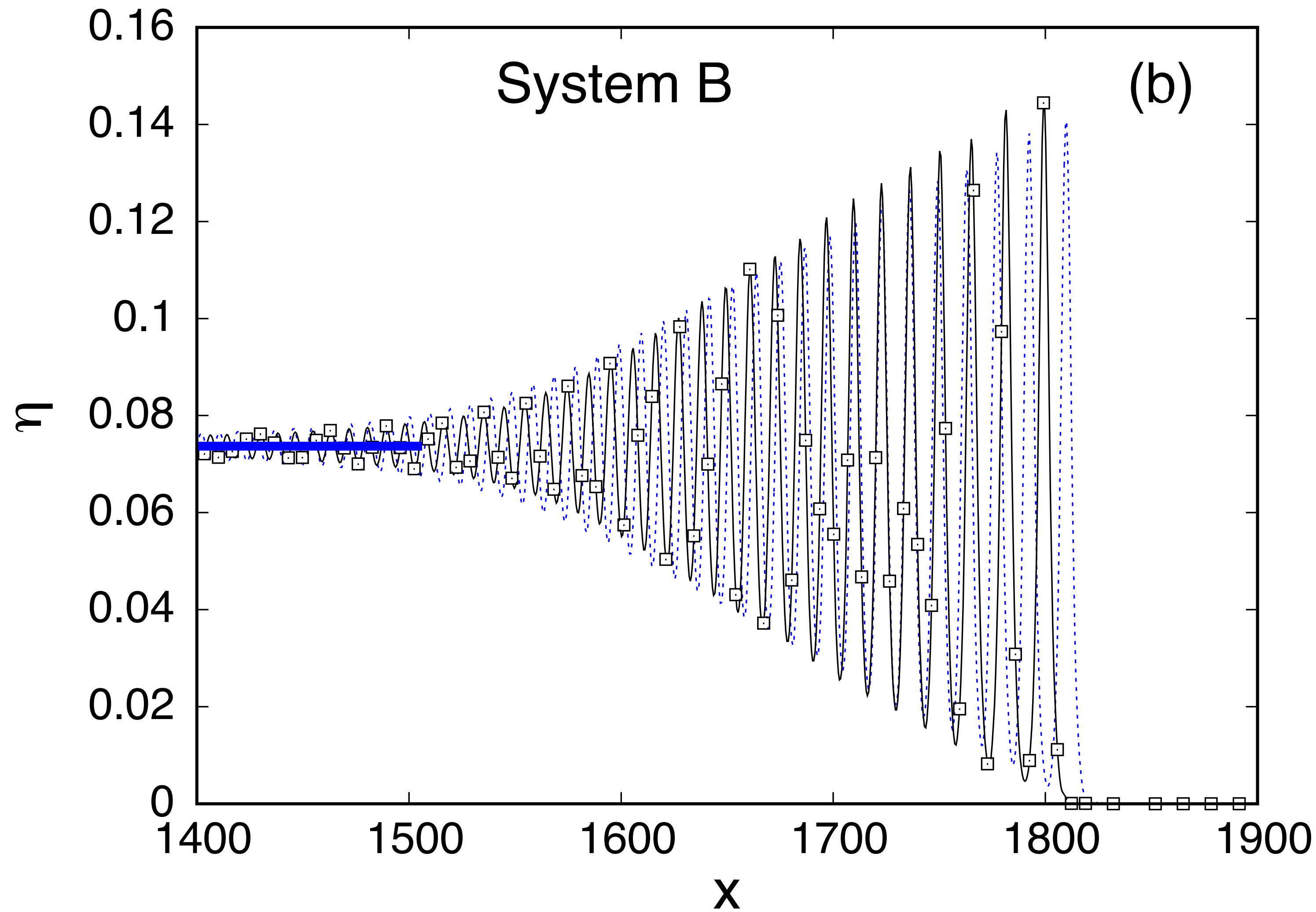
[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids*, 2021]

Sistema B: Boussinesq-Hamiltonian system

$$H = \frac{1}{2}\int_{\mathbb{R}} \xi((D)^2 - \frac{1}{3}(D)^4)\xi + \eta(\partial_x\xi)^2 + \eta^2)dx$$

$$\eta_t = -u_x - \frac{1}{3}u_{xxx} - (\eta u)_x$$

$$u_t = -\eta_x - uu_x.$$



$$V = 1 + \frac{1}{2}a,$$

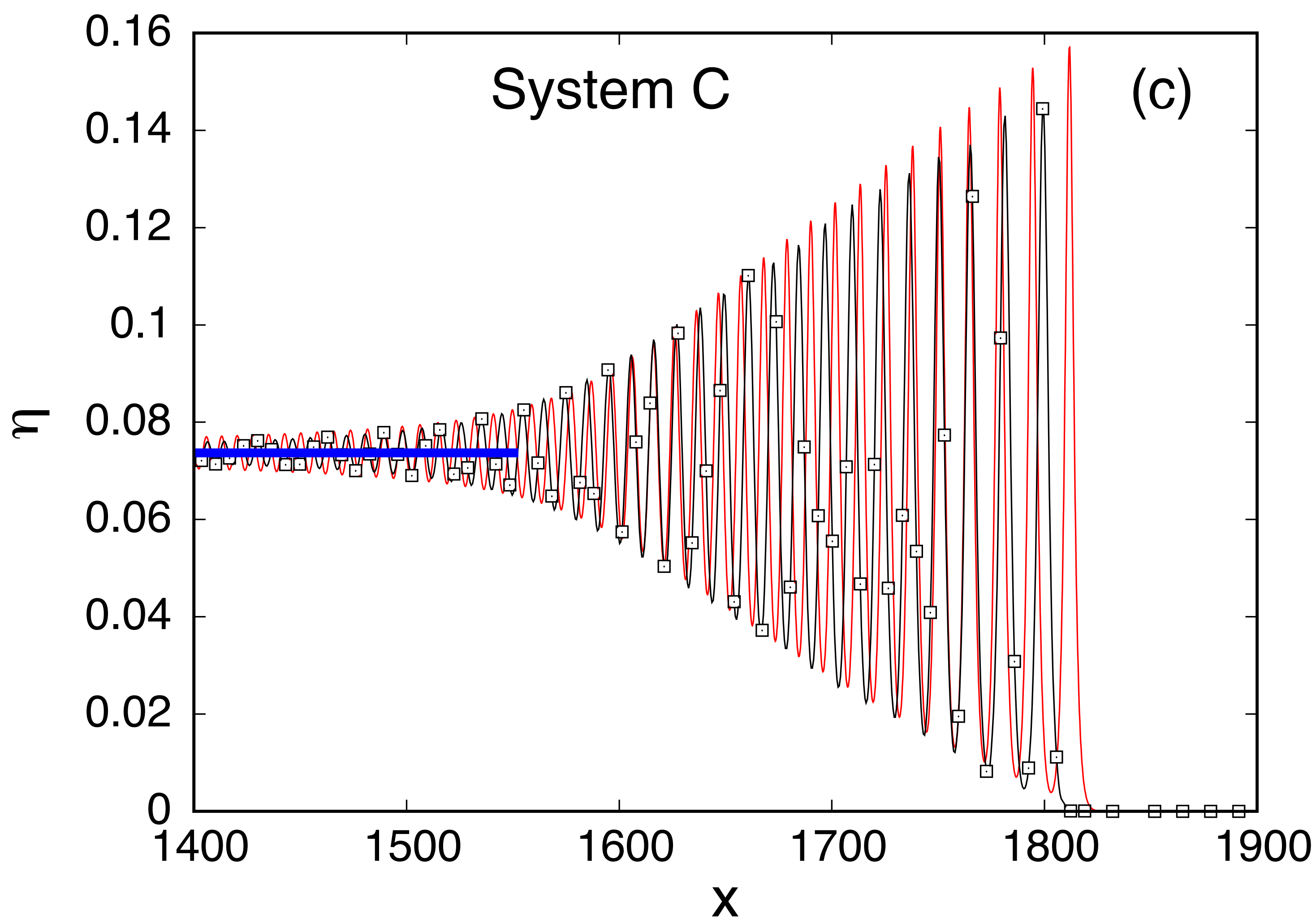
[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids,* 2021]

Sistema C: Full-dispersion shallow water equations

$$\eta_t = -u_x - (\eta u)_x,$$
$$u_t = -uu_x \boxed{- \eta_x - \frac{1}{3}\eta_{xxx}}$$

$$\eta_t = -u_x - (\eta u)_x,$$
$$u_t = -uu_x \boxed{- \partial_x \left(\left[\frac{\tanh D}{D} \right] \eta \right)}$$

[Hur-Tao, 2018]
[Hur-Padey, 2019]
[Carter, 2018]



[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids,* 2021]

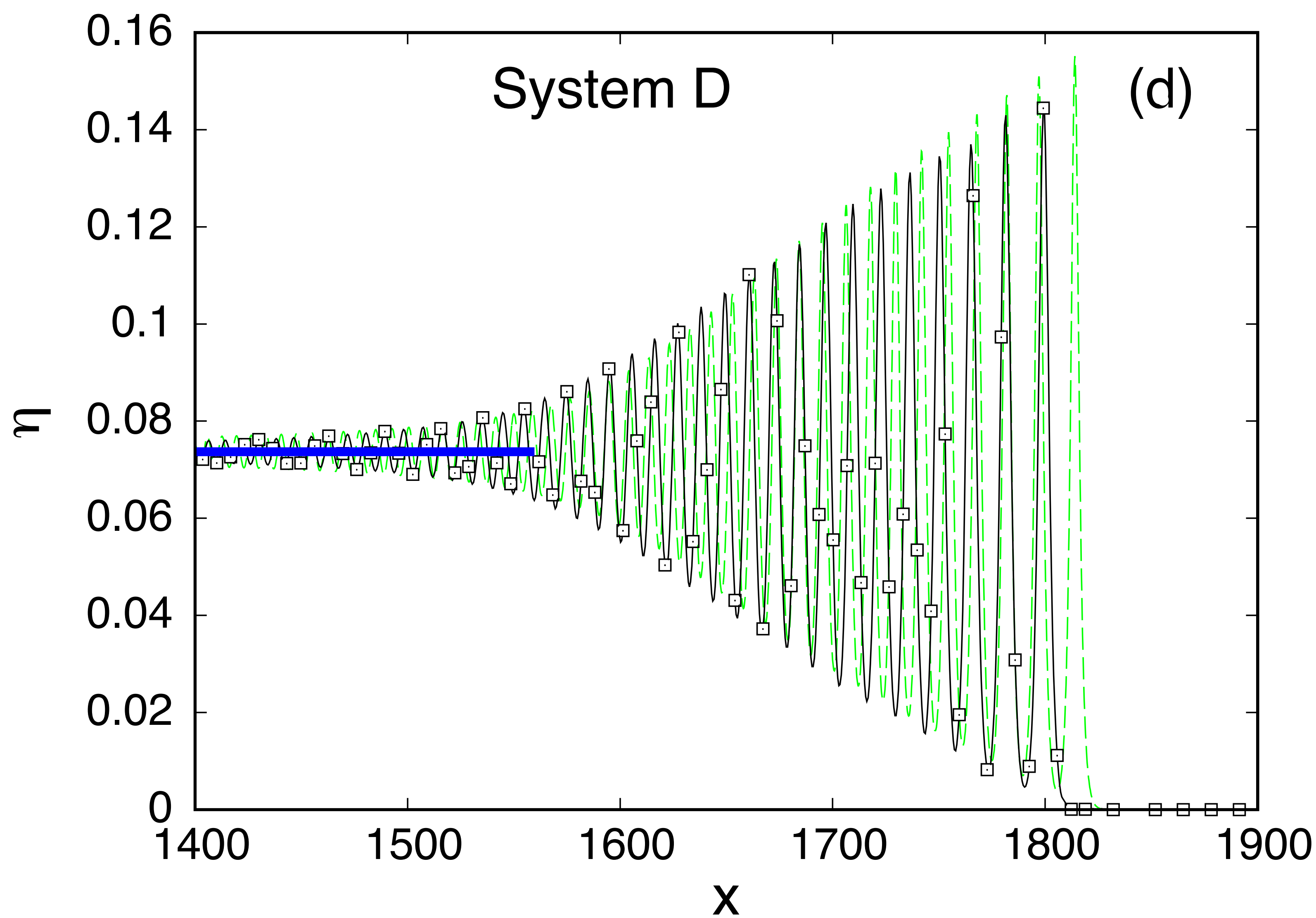
Sistema D: Whitham-Boussineq System

$$H = \frac{1}{2}\int_{\mathbb{R}} (\xi D \tanh(D)\xi + \eta(\partial_x \xi)^2 + \eta^2)dx$$

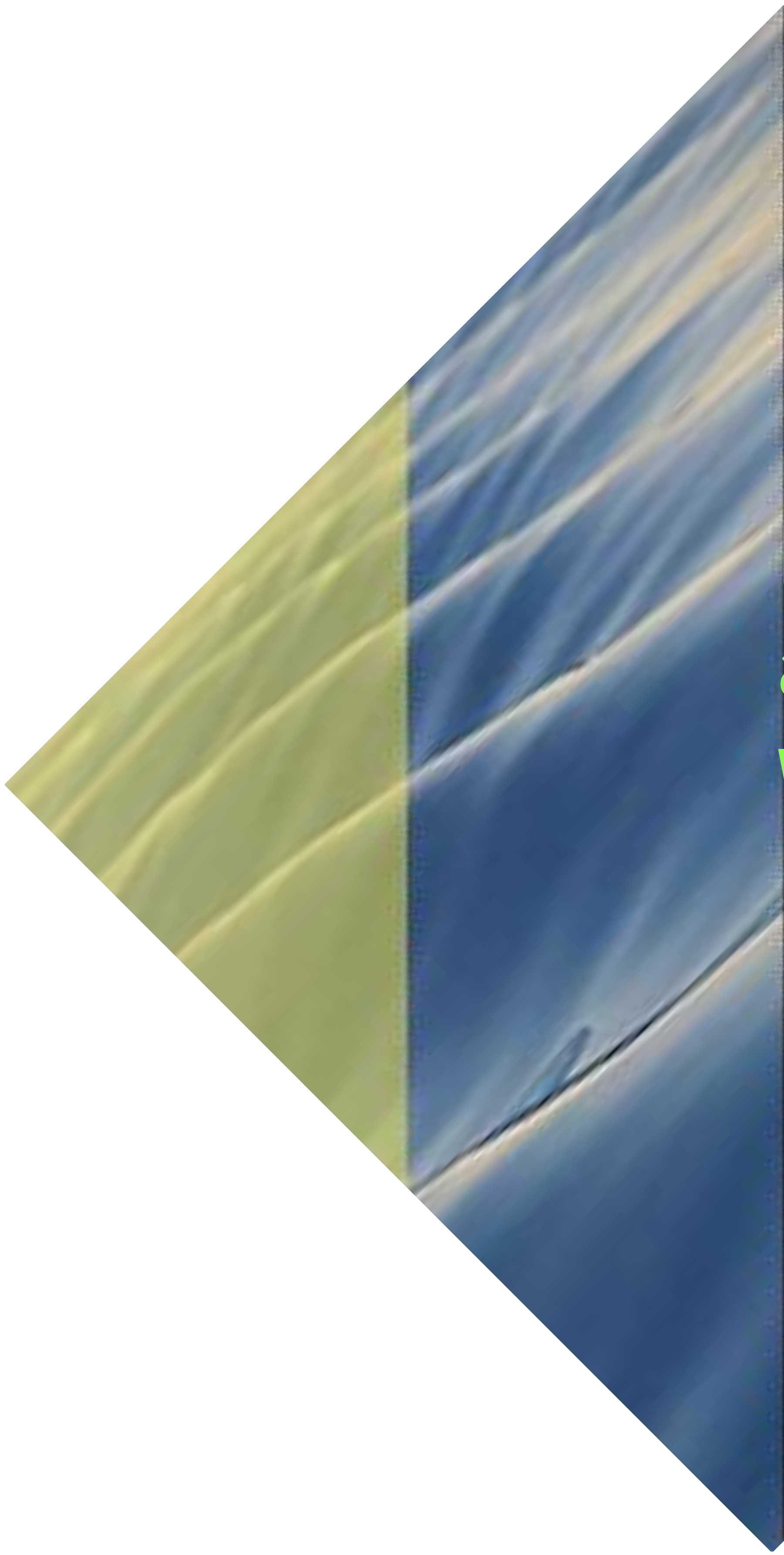
$$\eta_t = -\partial_x\left(\left[\frac{\tanh D}{D}\right]u\right) - (\eta u)_x,$$

$$u_t = -\eta_x - uu_x.$$

[Aceves-Panayotaros, Wave Motion, 2014]



[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids,* 2021]



DS-fitting Method
applied to
Weakly-Nonlinear
water waves models.

All Systems (A)–(D) have the same non-dispersive limit

$$\eta_t + u_x + (\eta u)_x = 0,$$

$$u_t + \eta_x + u u_x = 0.$$

$$\begin{pmatrix} \eta \\ u \end{pmatrix}_t = \begin{bmatrix} u & \eta + 1 \\ 1 & u \end{bmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix}_x$$

Diagonal form

$$\begin{pmatrix} \eta \\ u \end{pmatrix}_t = \begin{bmatrix} u + \sqrt{\eta + 1} & 0 \\ 0 & u - \sqrt{\eta + 1} \end{bmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix}_x$$

The hyperbolic system can be set in Riemann invariant form and this means that the corresponding Riemann invariants and characteristics velocities are:

$$u + 2\sqrt{\eta + 1} = R_+ \text{ on } \left(\frac{\partial x}{\partial t}\right)_+ = u + \sqrt{\eta + 1} = V_+$$

$$u - 2\sqrt{\eta + 1} = R_- \text{ on } \left(\frac{\partial x}{\partial t}\right)_- = u - \sqrt{\eta + 1} = V_-$$

Sistema A: Standard Boussinesq system

$$\omega^{System\,A}(\bar{\eta},k) = \bar{u}k + k\sqrt{(1+\bar{\eta})\left(1-\frac{1}{3}k^2\right)}$$

Sistema B: Boussinesq del Hamiltoniano

$$\omega^{System\,B} = \bar{u}k + k\left[1+\bar{\eta}-\frac{1}{3}k^2\right]^{1/2}$$

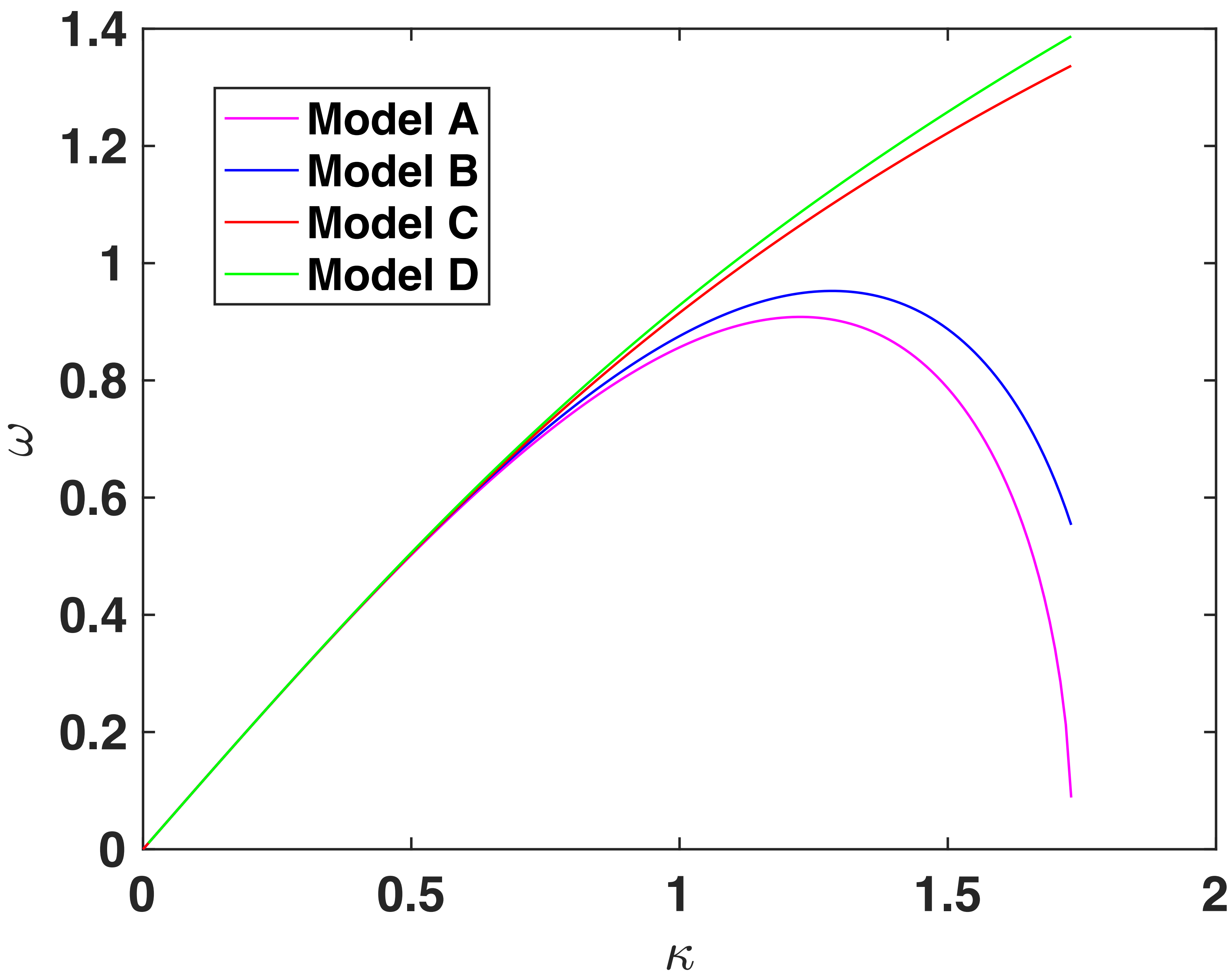
Sistema C: Full-dispersion shallow water equations

$$\omega^{System\,C} = \bar{u}k + \sqrt{(1+\bar{\eta})k\tanh k}.$$

Sistema D: Whitham-Boussineq System

$$\omega^{System\,D} = \bar{u}k + k\left[\frac{\tanh k}{k}+\bar{\eta}\right]^{1/2}$$

The linear dispersion relation ω(k) is real-valued



Sistema A: Standard Boussinesq system

$$\omega^{System\,A}(\bar{\eta},k) = \bar{u}k + k\sqrt{(1+\bar{\eta})\left(1-\frac{1}{3}k^2\right)}$$

$$\frac{dk}{d\bar{\eta}} = \frac{\frac{\partial \omega^{System\,A}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \omega^{System\,A}}{\partial k}} = \frac{k}{2(1+\bar{\eta})}\frac{2\sqrt{1-\frac{1}{3}k^2}+1-\frac{1}{3}k^2}{\sqrt{1-\frac{1}{3}k^2}-1+\frac{2}{3}k^2}$$

3.1 Dispersion relation from a background state $(\bar{\eta}, \bar{u})$ associated with the weakly non-linear models

- Dispersion relation for the Standard Boussinesq model

$$\omega^{SB}(\bar{\rho}, k) = 2(\sqrt{\bar{\rho}} - 1)k + k\sqrt{\bar{\rho}(1 - \frac{k^2}{3})} \tag{3.13}$$

(3.14)

where $\bar{\rho} = \bar{\eta} + 1$ And the conjugate dispersion relation is:

$$\begin{aligned}
\tilde{\omega}^{SB}(\bar{\rho}, \tilde{k}) &= -i\omega_{+}^{SB}(\bar{\rho}, i\tilde{k}) && (3.15)\\
&= -i[2(\sqrt{\bar{\rho}} - 1)(i\tilde{k}) + i\tilde{k}\sqrt{\bar{\rho}(1 - \frac{(i\tilde{k})^2}{3})}] && (3.16)\\
&= 2(\sqrt{\bar{\rho}} - 1)\tilde{k} + \tilde{k}\sqrt{\bar{\rho}(1 + \frac{\tilde{k}^2}{3})} && (3.17)
\end{aligned}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\frac{\partial \tilde{\omega}^{SystemA}}{\partial \bar{\eta}}}{V_{+} - \frac{\partial \tilde{\omega}^{SystemA}}{\partial \tilde{k}}} = \frac{\tilde{k}}{2(1+\bar{\eta})} \frac{2\sqrt{1 + \frac{1}{3}\tilde{k}^2} + 1 + \frac{1}{3}\tilde{k}^2}{\sqrt{1 + \frac{1}{3}\tilde{k}^2} - 1 - \frac{2}{3}\tilde{k}^2}$$

Sistema B: Boussinesq del Hamiltoniano

$$\omega^{System\,B} = \bar{u}k + k\left[1 + \bar{\eta} - \frac{1}{3}k^2\right]^{1/2}$$

$$\frac{dk}{d\bar{\eta}} = \frac{\frac{\partial \omega^{System\,B}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \omega^{System\,B}}{\partial k}} \quad = \quad \frac{k}{\bar{\rho}} \frac{(2\sqrt{1 - \frac{k^2}{3\bar{\rho}}} + 1)}{2\sqrt{1 - \frac{k^2}{3\bar{\rho}}} - 2(1 - \frac{k^2}{3\bar{\rho}}) + \frac{2k^2}{3\bar{\rho}}}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\frac{\partial \tilde{\omega}^{System\,B}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \tilde{\omega}^{System\,B}}{\partial \tilde{k}}} \quad = \quad \frac{\tilde{k}}{\bar{\rho}} \frac{(2\sqrt{1 + \frac{\tilde{k}^2}{3\bar{\rho}}} + 1)}{2\sqrt{1 + \frac{\tilde{k}^2}{3\bar{\rho}}} - 2(1 + \frac{\tilde{k}^2}{3\bar{\rho}}) - \frac{2\tilde{k}^2}{3\bar{\rho}}}$$

Sistema C: Full-dispersion shallow water equations

$$\omega^{System\,C} = \bar{u}k + \sqrt{(1+\bar{\eta})k\tanh k}.$$

$$\frac{dk}{d\bar{\eta}} = \frac{\sqrt{k\tanh k}}{1+\bar{\eta}} \frac{k + \frac{1}{2}\sqrt{k\tanh k}}{\sqrt{k\tanh k} - \frac{1}{2}\tanh k - \frac{1}{2}k\,\mathrm{sech}^2 k}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\sqrt{\tilde{k}\tan\tilde{k}}}{1+\bar{\eta}} \frac{\tilde{k} + \frac{1}{2}\sqrt{\tilde{k}\tan\tilde{k}}}{\sqrt{\tilde{k}\tan\tilde{k}} - \frac{1}{2}\tan\tilde{k} - \frac{1}{2}\tilde{k}\sec^2\tilde{k}}$$

Sistema D: Whitham-Boussineq System

$$\omega^{System\,D} = \bar{u}k + k\left[\frac{\tanh k}{k} + \bar{\eta}\right]^{1/2}$$

$$\frac{dk}{d\bar{\eta}} = \frac{k}{\sqrt{1+\bar{\eta}}}\,\frac{\sqrt{\frac{\tanh k}{k}+\bar{\eta}}+\frac{1}{2}\sqrt{1+\bar{\eta}}}{\sqrt{1+\bar{\eta}}\sqrt{\frac{\tanh k}{k}+\bar{\eta}}-\bar{\eta}-\frac{1}{2}\frac{\tanh k}{k}-\frac{1}{2}\operatorname{sech}^2 k}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\tilde{k}}{\sqrt{1+\bar{\eta}}}\,\frac{\sqrt{\frac{\tan \tilde{k}}{\tilde{k}}+\bar{\eta}}+\frac{1}{2}\sqrt{1+\bar{\eta}}}{\sqrt{1+\bar{\eta}}\sqrt{\frac{\tan \tilde{k}}{\tilde{k}}+\bar{\eta}}-\bar{\eta}-\frac{1}{2}\frac{\tan \tilde{k}}{\tilde{k}}-\frac{1}{2}\sec^2 \tilde{k}}$$

Sistema A: Standard Boussinesq system

$$\frac{dk}{d\bar{\eta}} = \frac{\frac{\partial \omega^{System A}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \omega^{System A}}{\partial k}} = \frac{k}{2(1+\bar{\eta})} \frac{2\sqrt{1-\frac{1}{3}k^2} + 1 - \frac{1}{3}k^2}{\sqrt{1-\frac{1}{3}k^2} - 1 + \frac{2}{3}k^2}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\frac{\partial \tilde{\omega}^{System A}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \tilde{\omega}^{System A}}{\partial \tilde{k}}} = \frac{\tilde{k}}{2(1+\bar{\eta})} \frac{2\sqrt{1+\frac{1}{3}\tilde{k}^2} + 1 + \frac{1}{3}\tilde{k}^2}{\sqrt{1+\frac{1}{3}\tilde{k}^2} - 1 - \frac{2}{3}\tilde{k}^2}$$

Sistema B: Boussinesq del Hamiltoniano

$$\frac{dk}{d\bar{\eta}} = \frac{\frac{\partial \omega^{System B}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \omega^{System B}}{\partial k}} = \frac{k}{\bar{\rho}} \frac{(2\sqrt{1-\frac{k^2}{3\bar{\rho}}} + 1)}{2\sqrt{1-\frac{k^2}{3\bar{\rho}}} - 2(1-\frac{k^2}{3\bar{\rho}}) + \frac{2k^2}{3\bar{\rho}}}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\frac{\partial \tilde{\omega}^{System B}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \tilde{\omega}^{System B}}{\partial \tilde{k}}} = \frac{\tilde{k}}{\bar{\rho}} \frac{(2\sqrt{1+\frac{\tilde{k}^2}{3\bar{\rho}}} + 1)}{2\sqrt{1+\frac{\tilde{k}^2}{3\bar{\rho}}} - 2(1+\frac{\tilde{k}^2}{3\bar{\rho}}) - \frac{2\tilde{k}^2}{3\bar{\rho}}}$$

Sistema C: Full-dispersion shallow water equations

$$\frac{dk}{d\bar{\eta}} = \frac{\sqrt{k\tanh k}}{1+\bar{\eta}} \frac{k + \frac{1}{2}\sqrt{k\tanh k}}{\sqrt{k\tanh k} - \frac{1}{2}\tanh k - \frac{1}{2}k\,\mathrm{sech}^2 k}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\sqrt{\tilde{k}\tan\tilde{k}}}{1+\bar{\eta}} \frac{\tilde{k} + \frac{1}{2}\sqrt{\tilde{k}\tan\tilde{k}}}{\sqrt{\tilde{k}\tan\tilde{k}} - \frac{1}{2}\tan\tilde{k} - \frac{1}{2}\tilde{k}\sec^2\tilde{k}}$$

Sistema D: Whitham-Boussineq System

$$\frac{dk}{d\bar{\eta}} = \frac{k}{\sqrt{1+\bar{\eta}}} \frac{\sqrt{\frac{\tanh k}{k} + \bar{\eta}} + \frac{1}{2}\sqrt{1+\bar{\eta}}}{\sqrt{1+\bar{\eta}}\sqrt{\frac{\tanh k}{k} + \bar{\eta}} - \bar{\eta} - \frac{1}{2}\frac{\tanh k}{k} - \frac{1}{2}\mathrm{sech}^2 k}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\tilde{k}}{\sqrt{1+\bar{\eta}}} \frac{\sqrt{\frac{\tan\tilde{k}}{\tilde{k}} + \bar{\eta}} + \frac{1}{2}\sqrt{1+\bar{\eta}}}{\sqrt{1+\bar{\eta}}\sqrt{\frac{\tan\tilde{k}}{\tilde{k}} + \bar{\eta}} - \bar{\eta} - \frac{1}{2}\frac{\tan\tilde{k}}{\tilde{k}} - \frac{1}{2}\sec^2\tilde{k}}$$

Main findings

	Velocities/ amplitudes of the Leading solitary wave	Leading soliton wave Amplitude/velocity relation
System A	$s_+ = \left[(1+\eta_+)\left(1+\frac{1}{3}\tilde{k}_+^2\right)\right]^{1/2},$ $a_+ = (1+\eta_+)\left(1+\frac{1}{3}\tilde{k}_+^2\right) - 1.$	$V = \sqrt{1+a}.$
System B	$s_+ = \left[1+\eta_+ + \frac{1}{3}\tilde{k}_+^2\right]^{1/2}$ $a_+ = 2\left[1+\eta_+ + \frac{1}{3}\tilde{k}_+^2\right]^{1/2} - 2.$	$V = 1 + \frac{1}{2}a,$
System C	Explicit ODE's equations/ numerical solutions using the second-order Runge–Kutta method.	$--$
System D	Explicit ODE's equations/ numerical solutions using the second-order Runge–Kutta method.	$--$

Performance of the Numerical Computation

► Boussinesq systems **A** and **B** and the Whitham–Boussinesq systems **C** and **D** were solved numerically using the pseudospectral method of Fornberg and Whitham as extended through the use of integrating factors to suppress high frequency instabilities.

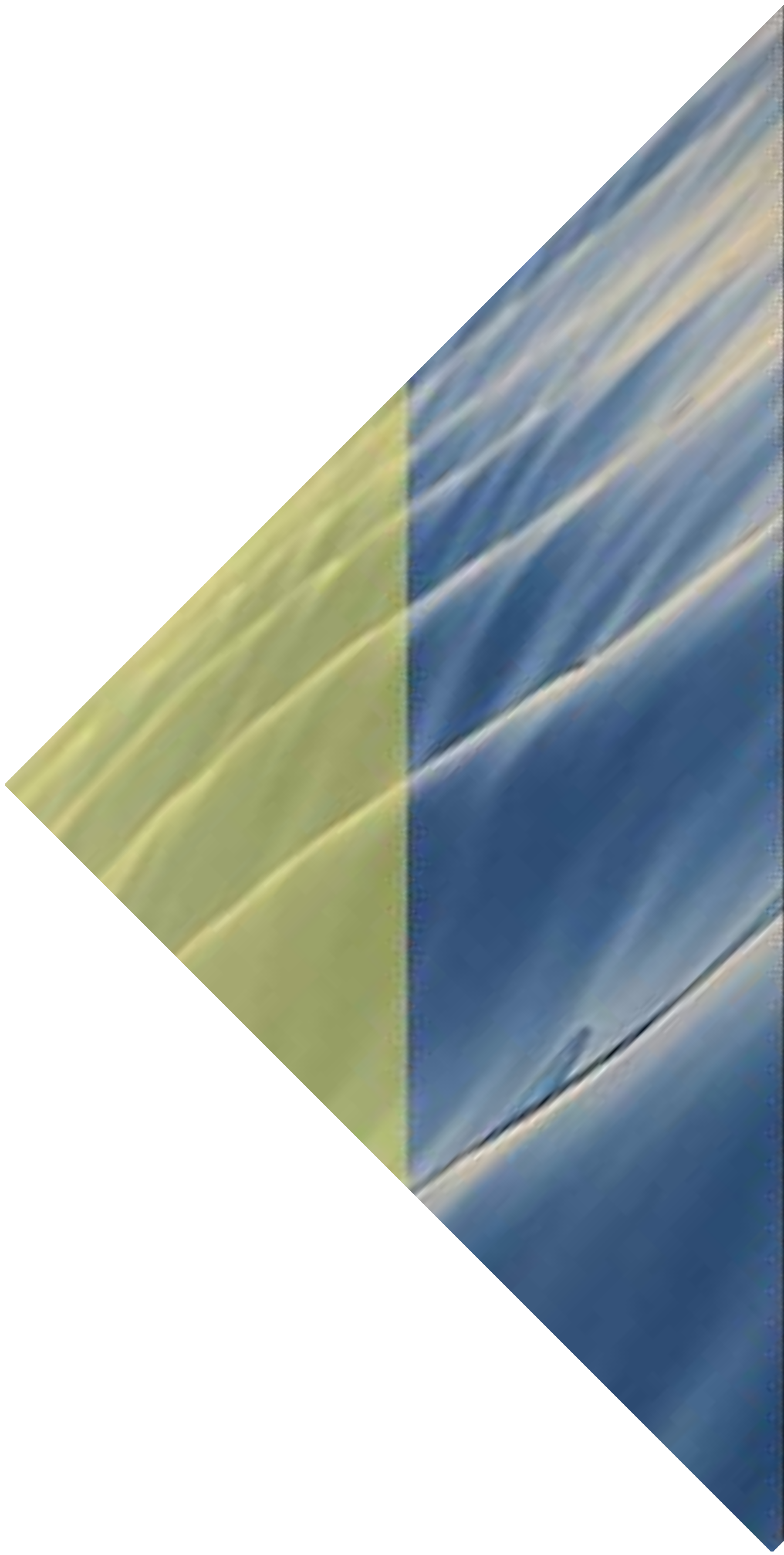
► These pseudospectral methods use the fast Fourier transform (FFT) to calculate the spatial derivatives, with the solution propagated forward in time using the second-order Runge–Kutta scheme and Strang splitting method.

► **n= 2^{16}** Fourier modes

► **Time step** $dt=0.005$

► The initial condition smoothed using hyperbolic tangents is $W=1$

$$\eta(x,0) = \eta_+ - \frac{1}{2}(\eta_- - \eta_+)\left[\tanh\frac{x}{W} - \tanh\frac{x-x_-}{W}\right].$$



Fully nonlinear **water** waves-***undular*** bores

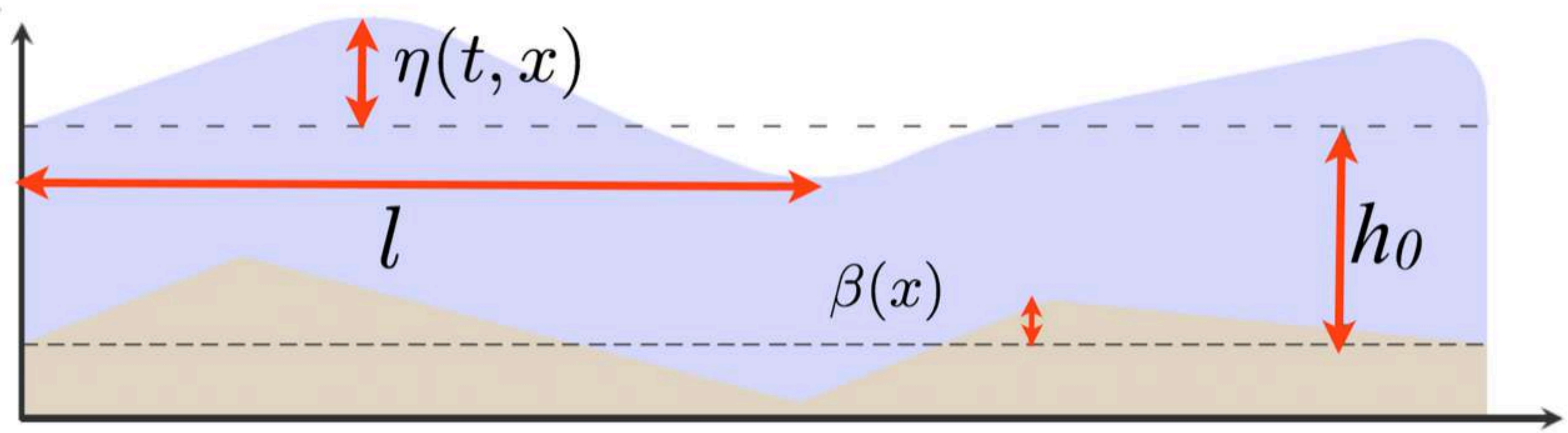
The water wave problem and the **Evolution equations.**

1. $$\varphi_{xx} + \varphi_{yy} = 0 \qquad \mathcal{D}(\beta(x), \eta(x,t))$$

2. $$\eta_t + \eta_x \varphi_x - \varphi_y = 0 \qquad y = \eta(x,t)$$

3. $$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2) + g\eta = 0 \qquad y = \eta(x,t)$$

4. $$\frac{\partial \varphi}{\partial n} = 0 \qquad y = -h_0 + \beta(x)$$



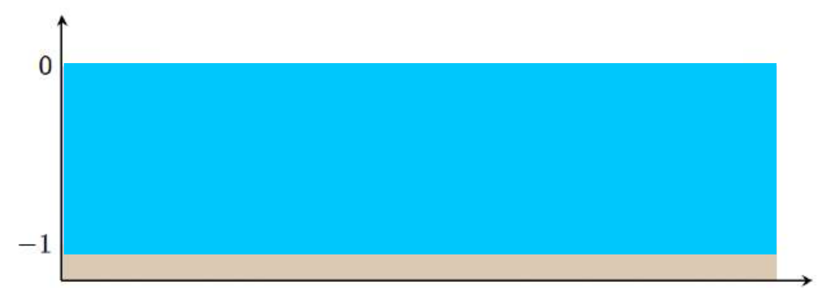
Remark: $$\eta_t = \varphi_y - \eta_x \varphi_x = \frac{\partial \varphi}{\partial n}$$

Hamiltonian Formulation of the water waves problem

$$H = \frac{1}{2}\int_{\mathbb{R}}(\xi G(\beta,\eta)\xi + g\eta^2)dx$$

$$\boxed{\begin{aligned} &\eta_t = G(\beta,\eta)\xi \\ &\xi_t = \tfrac{-1}{2(1+\eta_x^2)}(\xi_x^2 - (G(\beta,\eta)\xi)^2 - 2\eta_x\xi_x G(\beta,\eta)\xi)^2 - 2\eta_x\xi_x G(\beta,\eta)\xi) - g\eta \end{aligned}}$$

- In general there is not an explicit expression for it!
- This case gives rise to an explicit expression to de DN operator:



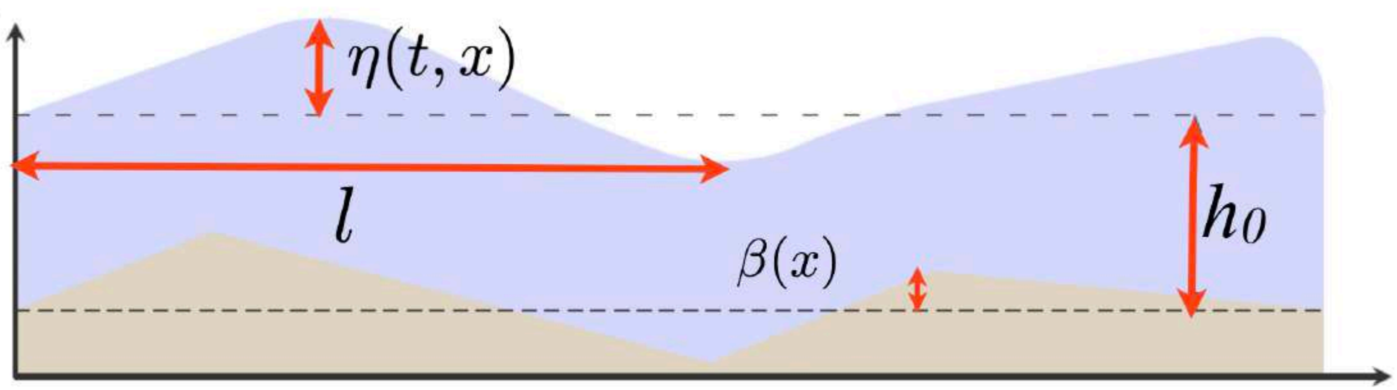
$$[G(0,0)]: \quad \xi \longmapsto D\tanh(D)\xi$$

What is this object?

Where D is as usual the operator $D = -i\partial_x$

$$\xi(x) \longmapsto \hat{\xi}(\kappa) \longmapsto \kappa\tanh(h_0\kappa)\hat{\xi}(\kappa) \longmapsto \int_{\mathbb{R}} \kappa\tanh(h_0\kappa)\hat{\xi}(\kappa)e^{i\kappa x}d\kappa := [D\tanh(h_0 D)]\xi$$

Series Expansion of **D-N operator**



$\delta = \frac{h_0^2}{l^2} \ll 1 \qquad \sim \qquad \epsilon = \frac{\eta}{h_0}$ **small!**

$$[G(\beta,\epsilon\eta)](\xi) = \sum_{n=0}^{\infty}[G_n(\beta,\eta)](\xi)\epsilon^n$$

$$\begin{aligned} G_0(\beta,\eta) &= D\tanh(h_0 D) + DL(\beta), \\ G_1(\beta,\eta) &= D\eta D - G_0\eta G_0, \\ G_2(\beta,\eta) &= \frac{1}{2}(G_0 D\eta^2 D - D^2\eta^2 G_0 - 2G_0\eta G_1), \end{aligned}$$

Coifman and Meyer (1985)
Craig, Schanz and Sulem (1997)
Craig and Sulem (2005)
Craig, Guyenne, Nicholls, Sulem (2005)

With $D = -i\partial_x$

and $L(\beta)$ Involve **pseudo-differential** operators

The **numerical solutions** of the water wave equations were found using the Hamiltonian formulation and based on a **pseudospectral method**

$$\mathbf{v}_t = \mathscr{L}\mathbf{v} + \mathscr{N}\mathbf{v}, \text{ where}$$

$$\mathscr{L} = \begin{pmatrix} 0 & G_0 \\ -g & 0 \end{pmatrix}, \quad \mathscr{N} = \begin{pmatrix} (G - G_0)\xi \\ -\dfrac{\left[\xi_x^2 - (G\xi)^2 - 2\xi_x\eta_x G\xi\right]}{2\left(1 + \eta_x^2\right)} \end{pmatrix},$$

$$\mathbf{v} = (\eta, \xi)^T$$

► The system is solved using a split step method where the linear andd nonlinear components are solved separately.

$$\mathbf{v}_t = \mathscr{L}\mathbf{v}, \qquad \mathbf{v}_t = \mathscr{N}\mathbf{v}$$

► The second order string splitting step method is then used for each time step.

$$\mathbf{v}^{n+1} = \mathscr{N}_{\frac{\Delta t}{2}} \mathscr{L}_{\Delta t} \mathscr{N}_{\frac{\Delta t}{2}} \mathbf{v}^n$$

Performance of the Numerical Computation

► **n= 2^{15}** Fourier modes

► **Time step** dt=0.01

► For the numerical solutions of the water wave equations, the domain lengths 10 000 or 11 000 were used.

► The initial condition smoothed using hyperbolic tangents is

$$\eta(x,0) = \eta_+ - \frac{1}{2}(\eta_- - \eta_+)\left[\tanh\frac{x}{W} - \tanh\frac{x - x_-}{W}\right].$$

Technical findings

► Three terms in series, up to $\mathbf{G_2}$, provide sufficient accuracy for these initial conditions.

► For **n= 2^{15}** 5 filtering the higher modes corresponding to wavenumbers lying in the band **[n/8,n/2]** was adequate to suppress numerical instability, particularly for higher initial jumps.

► There was some change in the solution as the smoothing width W varied from 10 to 5. Furthermore, **as W decreased from 5 to 2, there was no change** in the leading edge of the bore and negligible change at the trailing edge which was below the graphical accuracy of the figures.

$$\eta(x,0) = \eta_+ - \frac{1}{2}(\eta_- - \eta_+)\left[\tanh\frac{x}{W} - \tanh\frac{x - x_-}{W}\right].$$

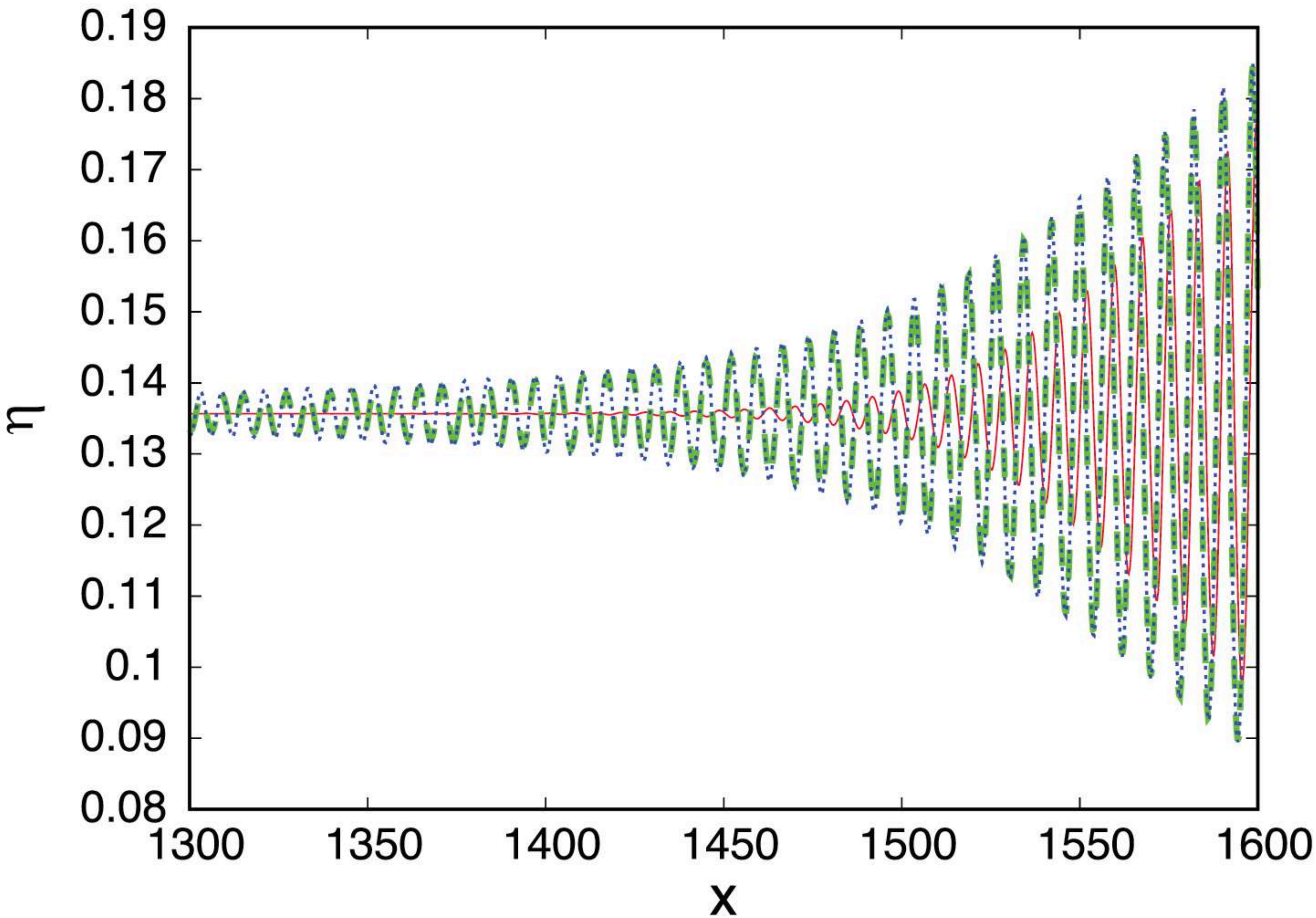
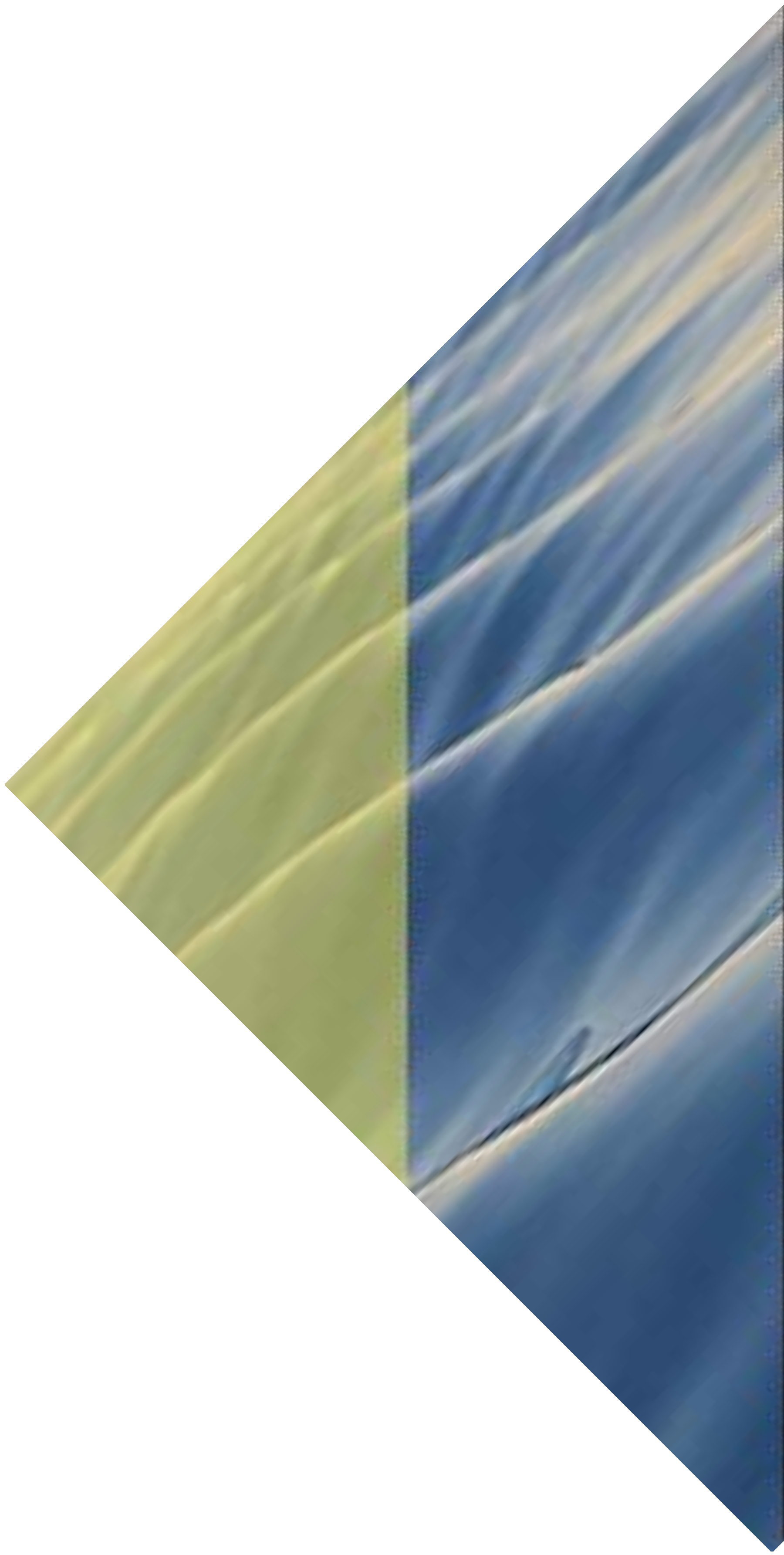
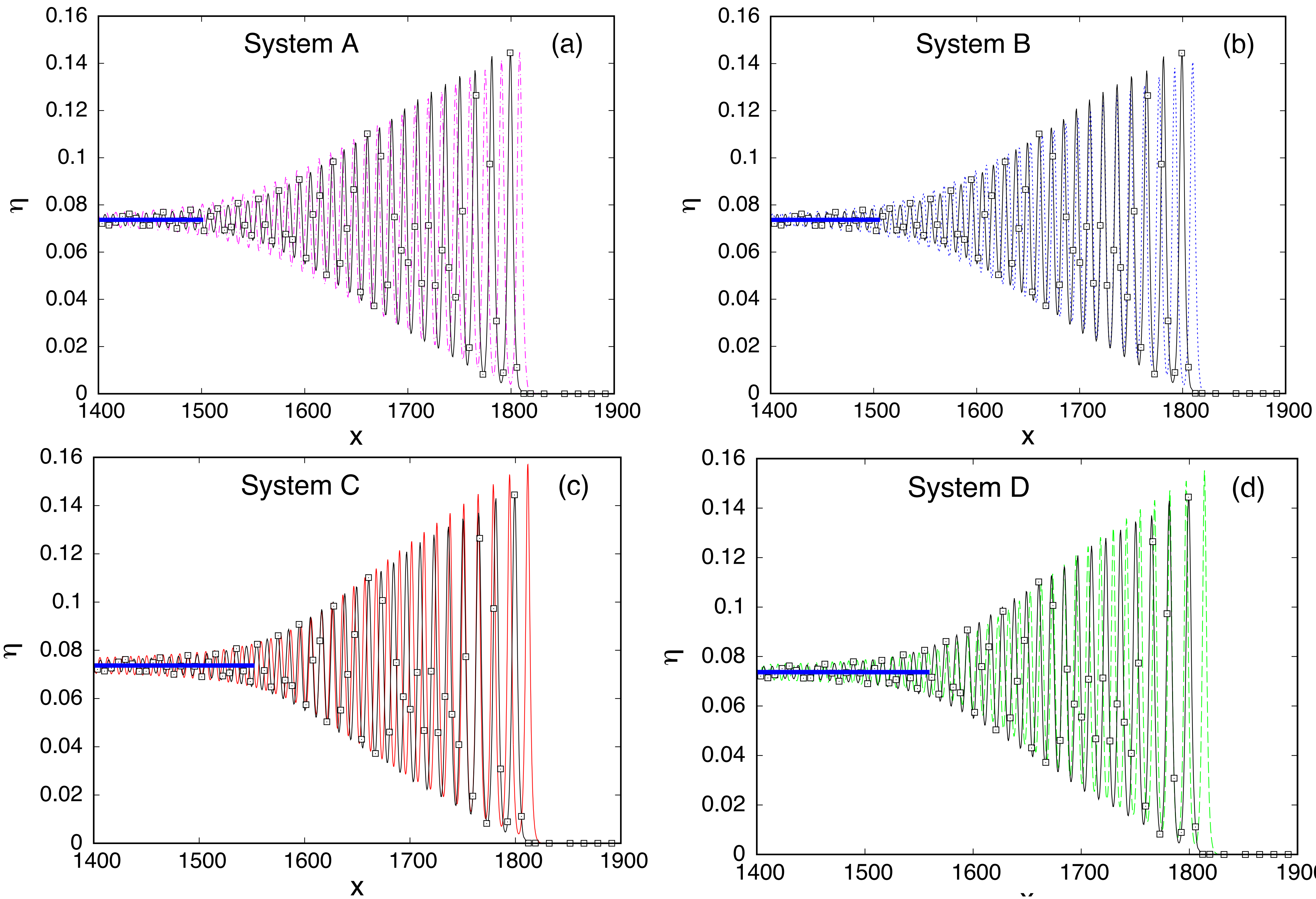


Figure shows that large smoothing width W=10 (red line) gives a greatly truncated trailing edge in comparison with the sharper initial conditions with W=2 (green dashed-line)and W=5/3 (blue-dotted line).



Results

Undular Bore evolution



Thik Line blue
The position of
the trailing
edge of the
undular bore

FIG. 4. Stable bore solutions of Boussinesq systems A (27) and B (28), the Whitham–Boussinesq systems C (29) and D (30) and the water wave equations (2)–(5) for $\Delta = 0.15$ at $t = 1700$. (a) System A: pink (dotted-dashed) line; water wave equations: black (full) line with squares. (b) System B: blue (dotted) line, water wave equations: black (full) line with squares. (c) System C: red (full) line; water wave equations: black (full) line with squares. (d) System D: green (dashed) line; water wave equations: black (solid) line with squares. Intermediate level (43): thick blue line.

[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids*, 2021]

Main Findings

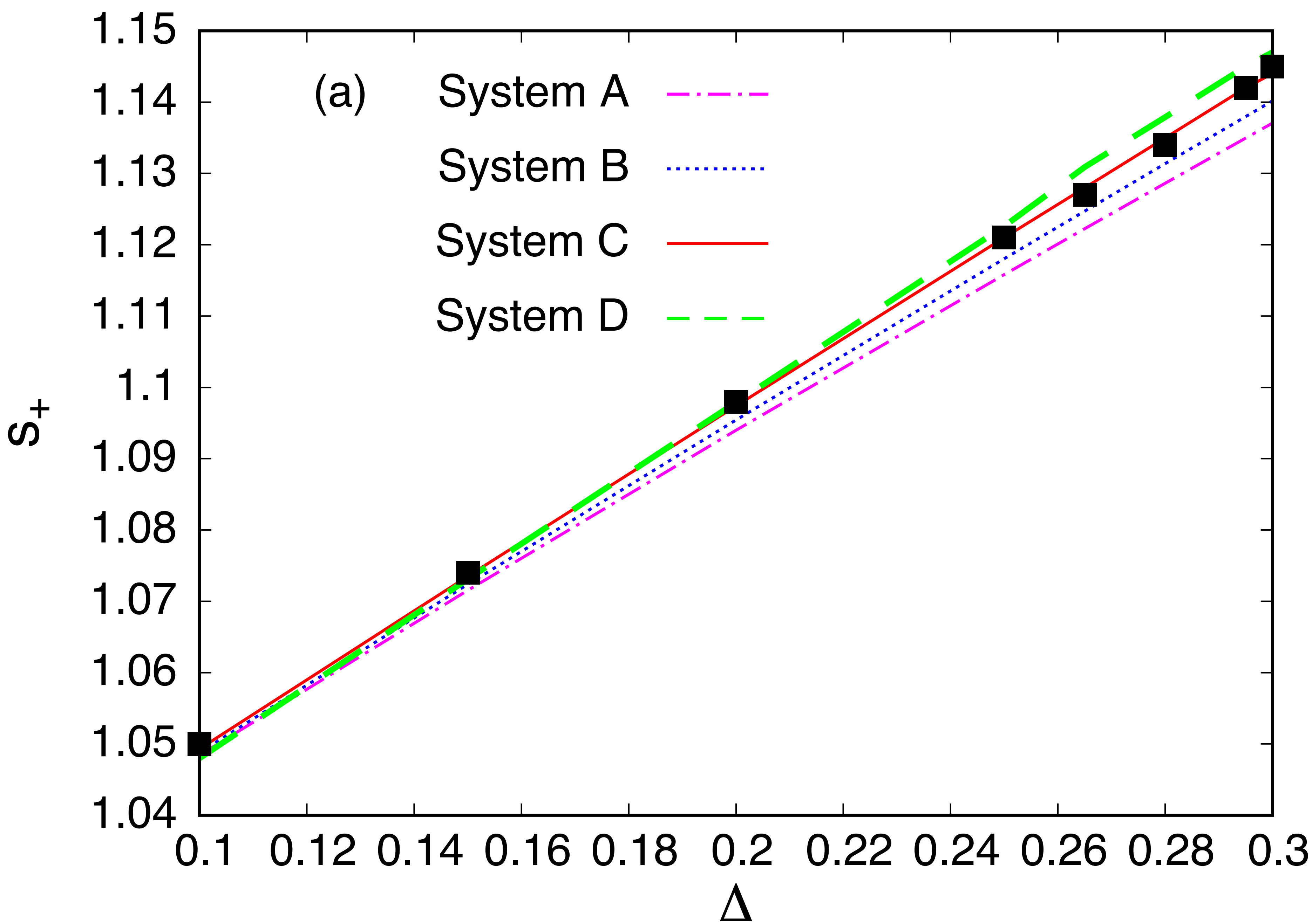
- The main difference between the water wave bore and the Boussinesq and Whitham–Boussinesq bores is a phase difference, with the water wave bore behind these weakly nonlinear bores.

- **All four weakly nonlinear systems give a good prediction** for the location of the **trailing edge** of the water wave bore.

- The trailing **edge group velocity for Systems A and B** is slightly lower than that for Systems C and D, so that t**he trailing edge position for Systems A and B corresponds to lower amplitude waves** of the water wave bore

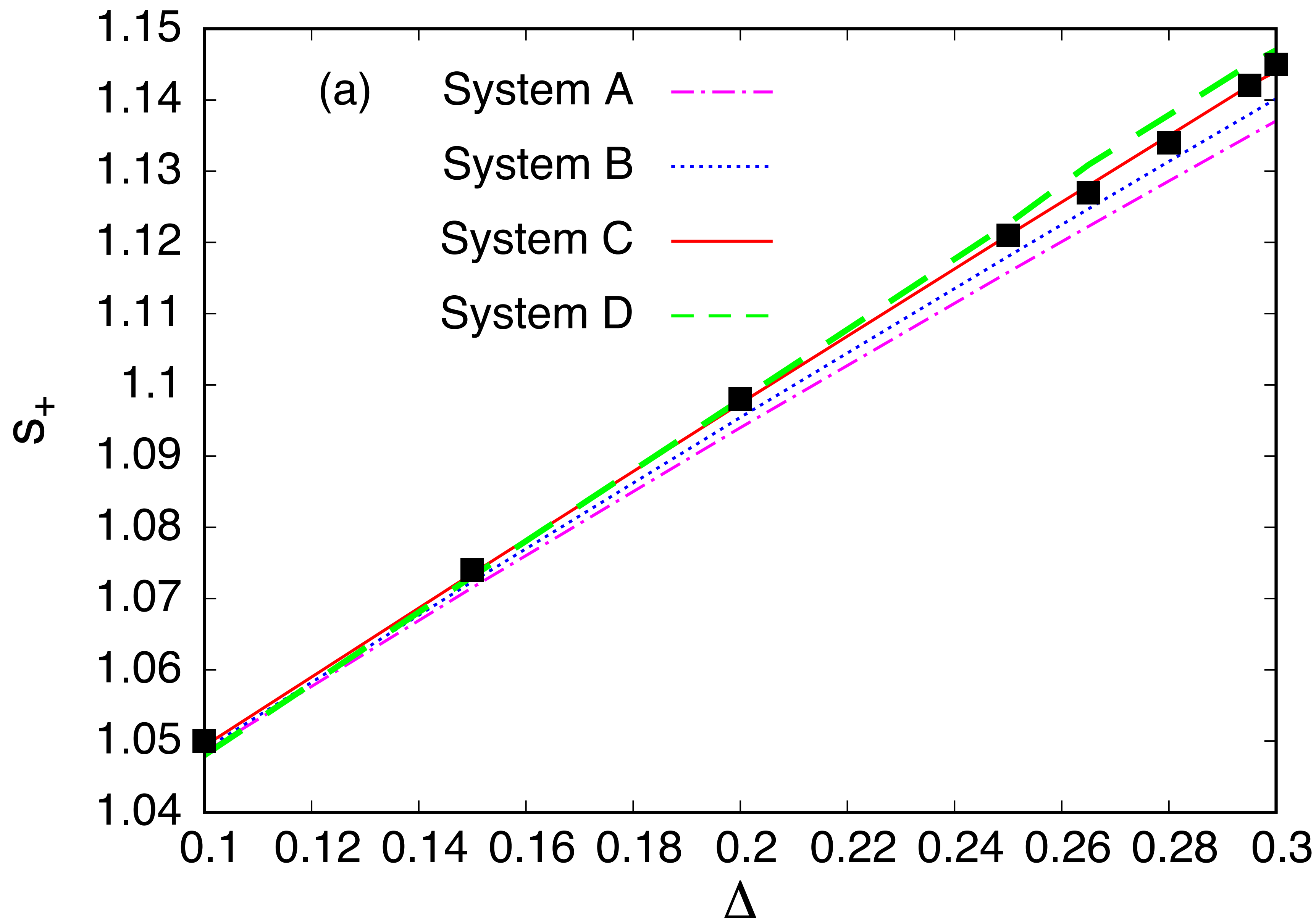
- It was found that **the numerical bore solution of the water wave equations** took an unrealistic amount of time to reach a steady state

Velocities of the **leading edge of the bore**



[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids*, 2021]

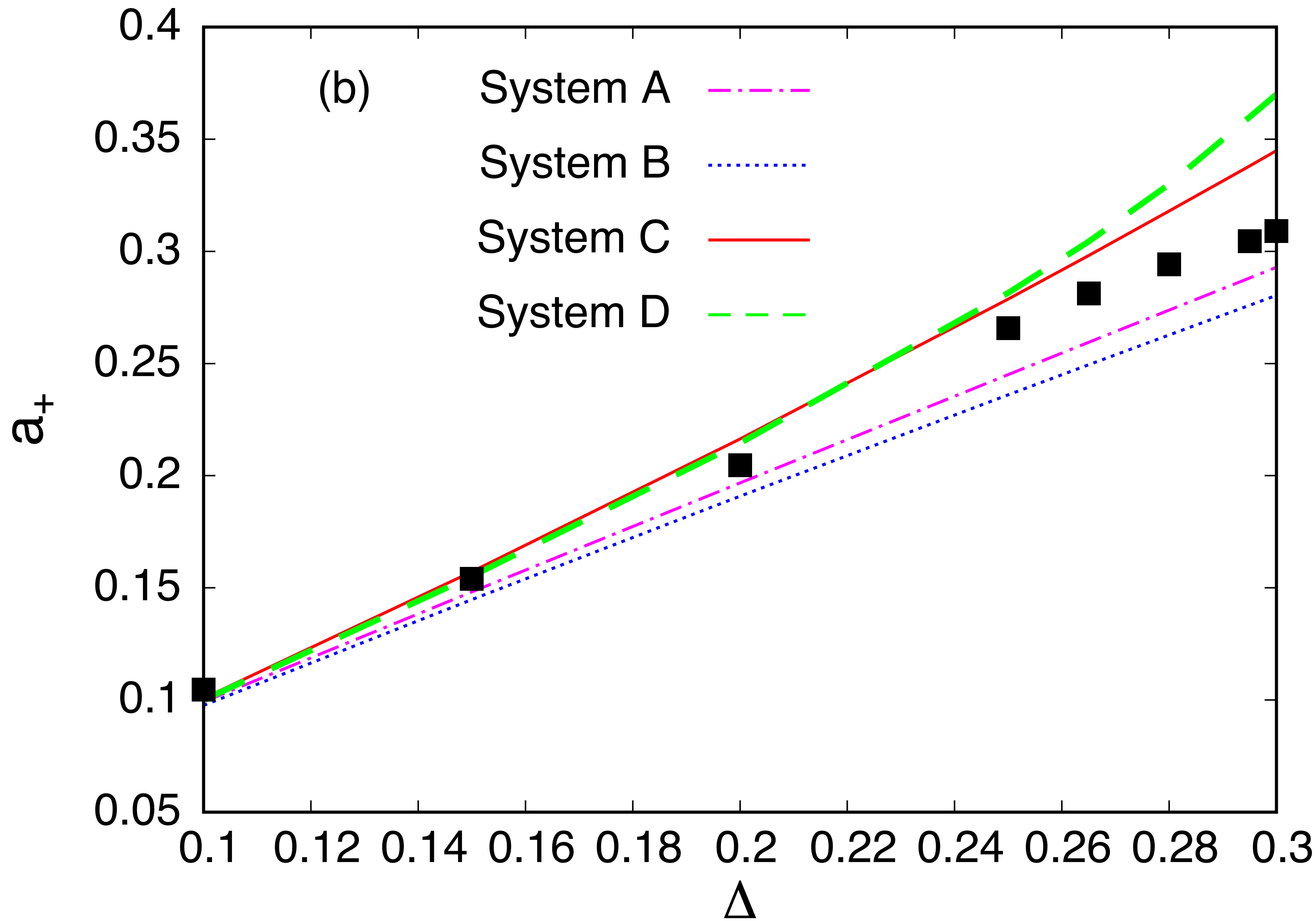
Main Findings



► The comparison starts at the jump height **Δ = 0.1 as below** this the **all systems give identical results to graphical accuracy.**

► Figure shows that **there is excellent agreement between solutions of the Whitham- Boussinesq systems** and numerical solutions of the water wave equations for the velocity of the leading edge of the bore. The maximum difference for System A being 0.7% and for System B 0.4%.

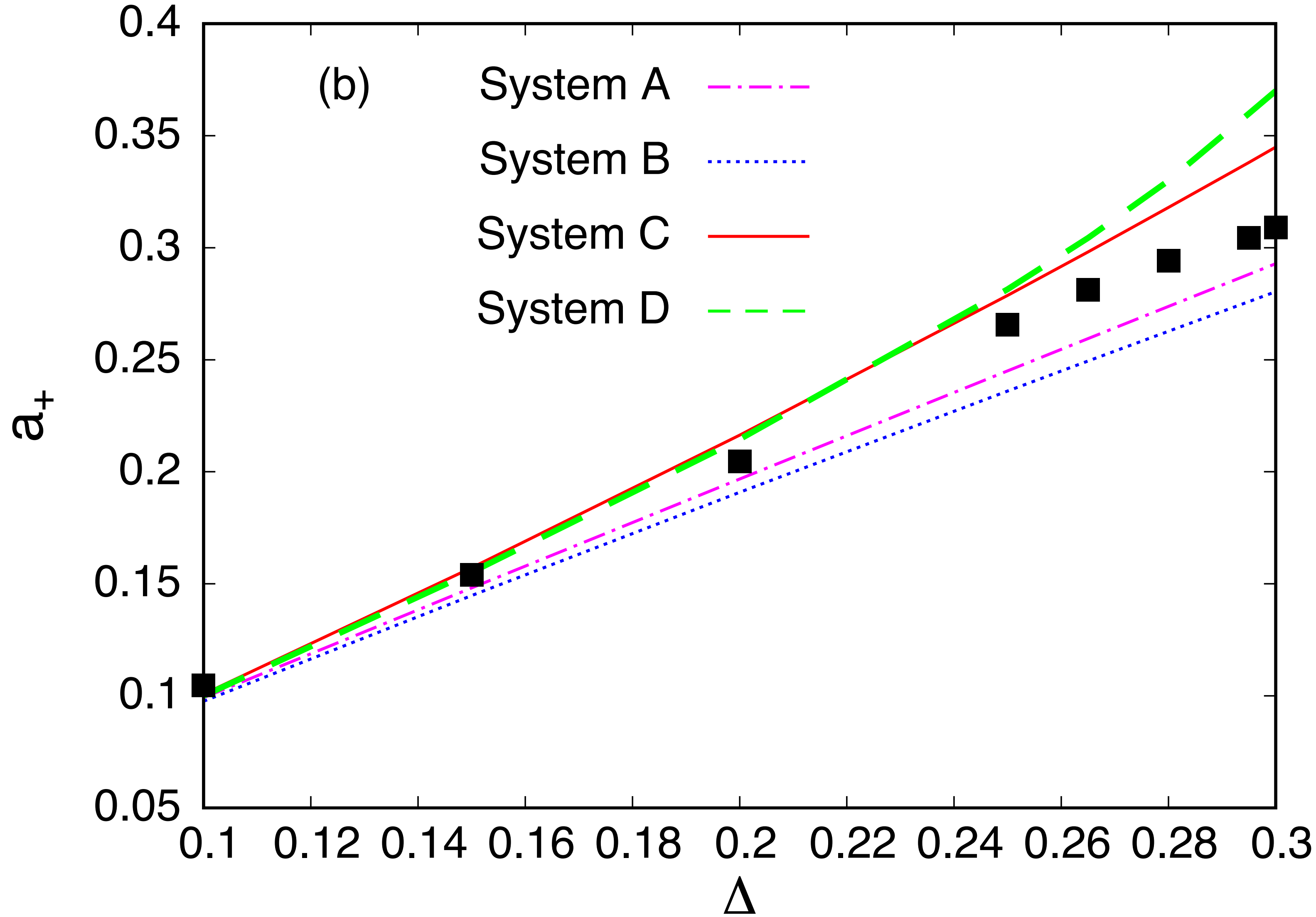
Amplitudes of the Leading solitary wave



[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids*, 2021]

Main Findings

► **The agreement between the solutions of the Boussinesq systems and the solution of the water wave equations is not as good** as for the leading edge velocity. But this is typical for bore solutions of nonlinear dispersive wave equations.

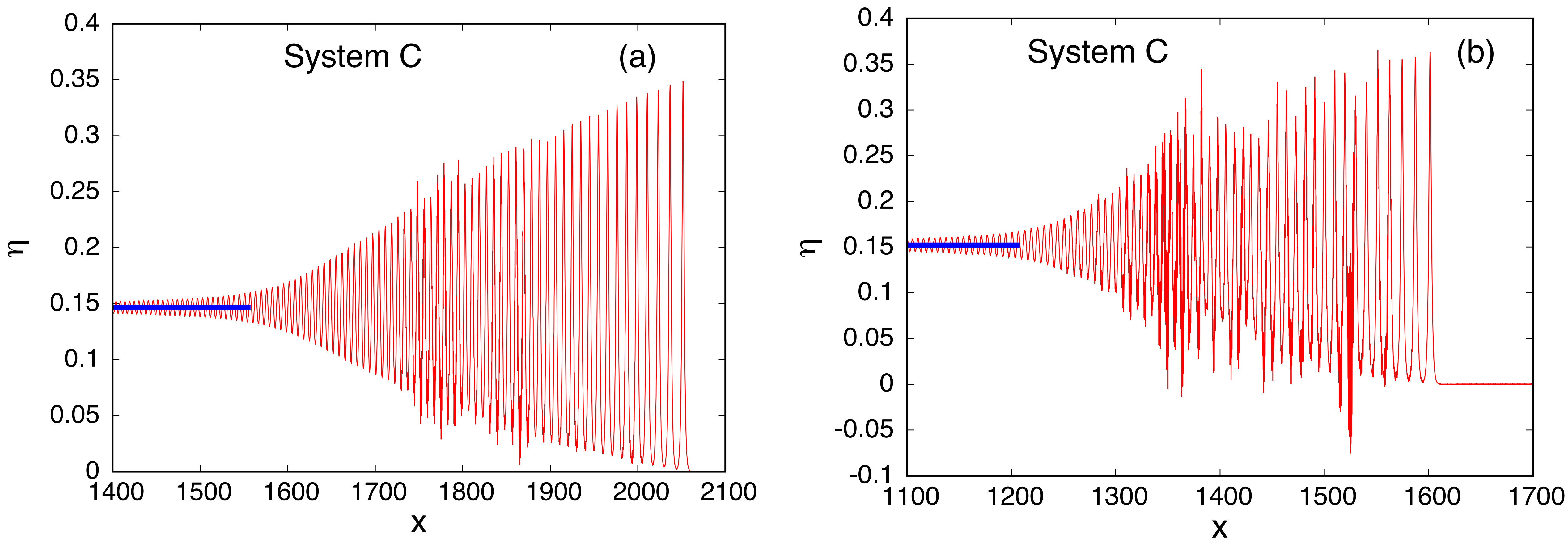


► The amplitude as given by the Whitham–Boussinesq systems grows much more rapidly as the jump height Δ increases toward the onset of instability around.

► The comparison shows that **the amplitude** of the bore as given by **the water wave equations lies between the amplitude as given by the Boussinesq and Whitham–Boussinesq systems,** but is **closer to the amplitude given by the Whitham–Boussinesq** systems.

The original motivation for the introduction of Whitham-type equations was that the inclusion of the full linear dispersion relation introduces short wave effects which are not present in equations on systems A and B

Sistema C: Full-dispersion shallow water equations



[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids,* 2021]

FIG. 5. Solutions of Whitham–Boussinesq systems C (29) and D (30) and the water wave equations (2)–(5). (a) System C: red (full) line at $t = 1800$ for $\Delta = 0.303$, (b) System C: red (full) line at $t = 1400$ for $\Delta = 0.315$, (c) System D: green (dashed) line at $t = 1700$ for $\Delta = 0.315$, (d) System D: green (dashed) line at $t = 1700$ for $\Delta = 0.32$, (e) water wave equations: black (solid) line at $t = 2200$ for $\Delta = 0.36$, and (f) water wave equations: black (solid) line at $t = 2400$ for $\Delta = 0.37$. Intermediate level (43): thick blue line.

Sistema D: Whitham-Boussineq System

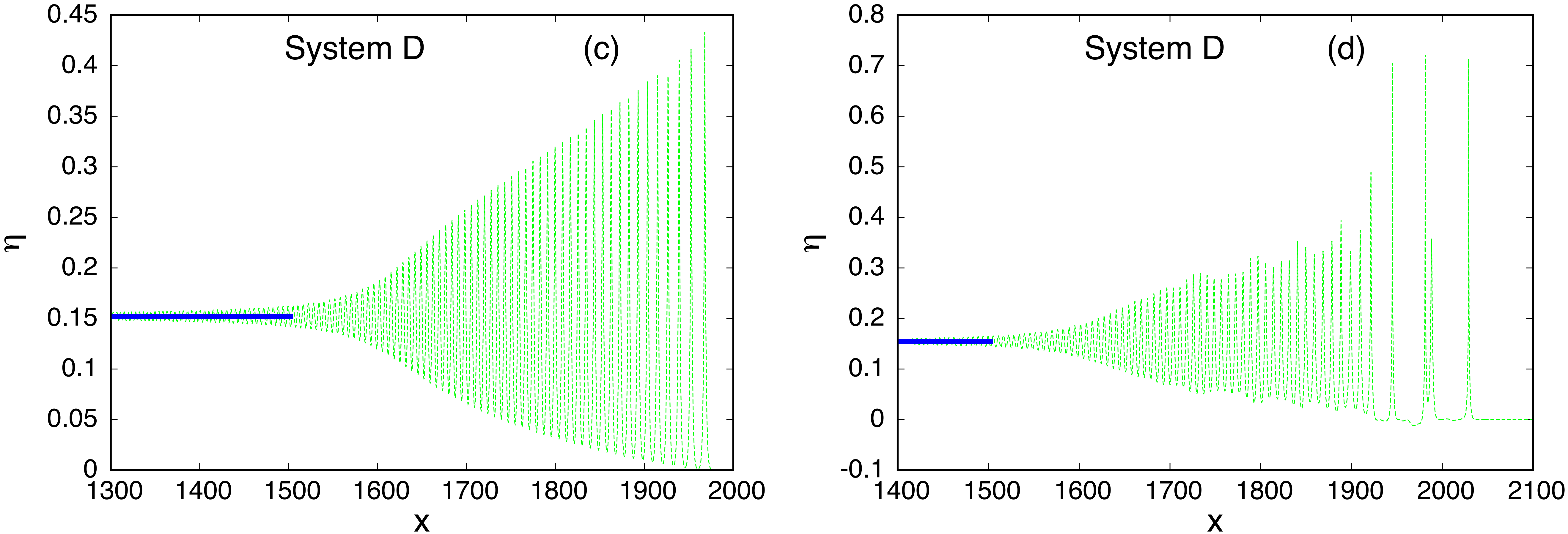


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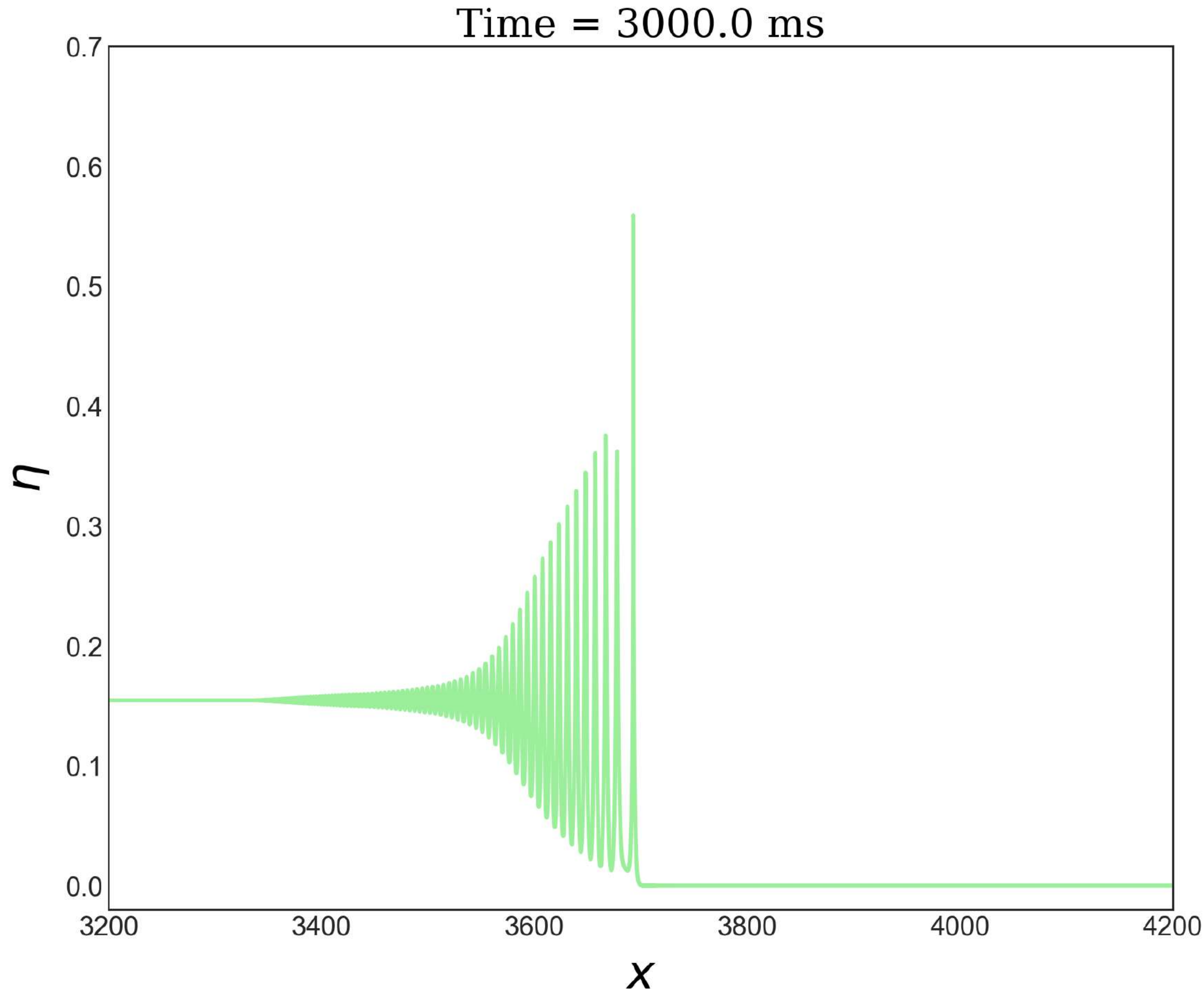
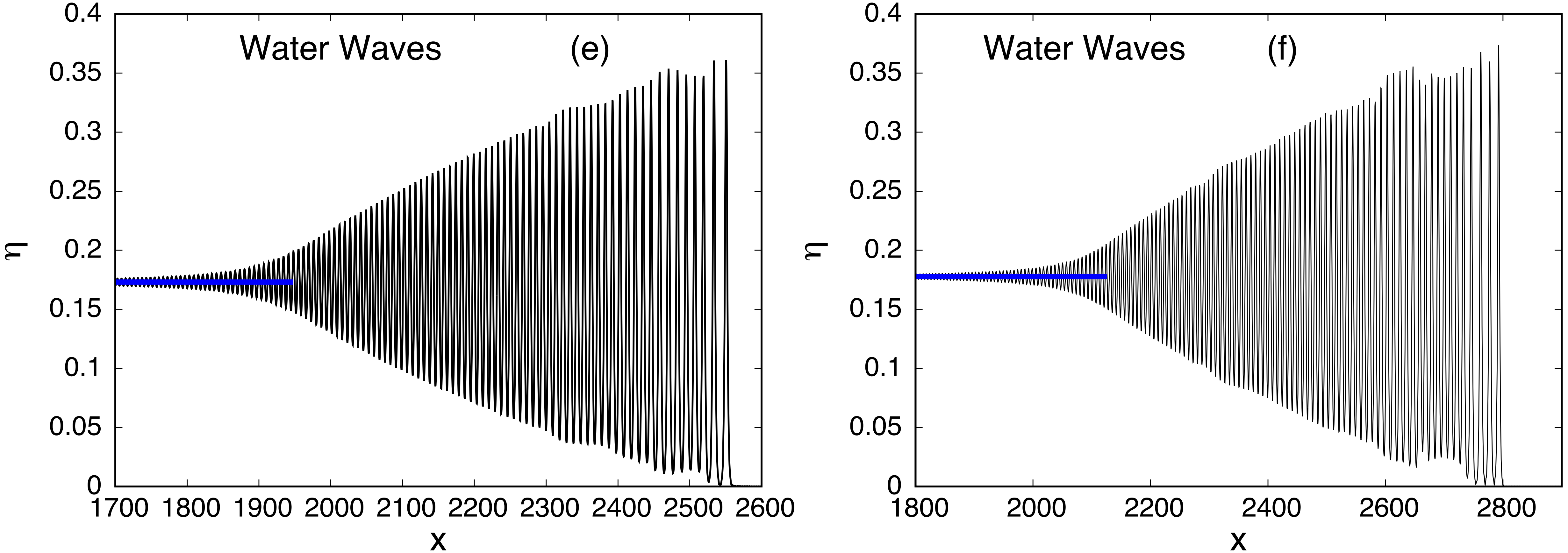
FIG. 5. Solutions of Whitham–Boussinesq systems C (29) and D (30) and the water wave equations (2)–(5). (a) System C: red (full) line at $t = 1800$ for $\Delta = 0.303$, (b) System C: red (full) line at $t = 1400$ for $\Delta = 0.315$, (c) System D: green (dashed) line at $t = 1700$ for $\Delta = 0.315$, (d) System D: green (dashed) line at $t = 1700$ for $\Delta = 0.32$, (e) water wave equations: black (solid) line at $t = 2200$ for $\Delta = 0.36$, and (f) water wave equations: black (solid) line at $t = 2400$ for $\Delta = 0.37$. Intermediate level (43): thick blue line.


Time = 3000.0 ms
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
η
3200
3400
3600
3800
4000
4200
x

Fully non-linear water waves equations



[Marchant -Smyth-Vargas-Magaña, *Physics of Fluids*, 2021]

FIG. 5. Solutions of Whitham–Boussinesq systems C (29) and D (30) and the water wave equations (2)–(5). (a) System C: red (full) line at $t = 1800$ for $\Delta = 0.303$, (b) System C: red (full) line at $t = 1400$ for $\Delta = 0.315$, (c) System D: green (dashed) line at $t = 1700$ for $\Delta = 0.315$, (d) System D: green (dashed) line at $t = 1700$ for $\Delta = 0.32$, (e) water wave equations: black (solid) line at $t = 2200$ for $\Delta = 0.36$, and (f) water wave equations: black (solid) line at $t = 2400$ for $\Delta = 0.37$. Intermediate level (43): thick blue line.

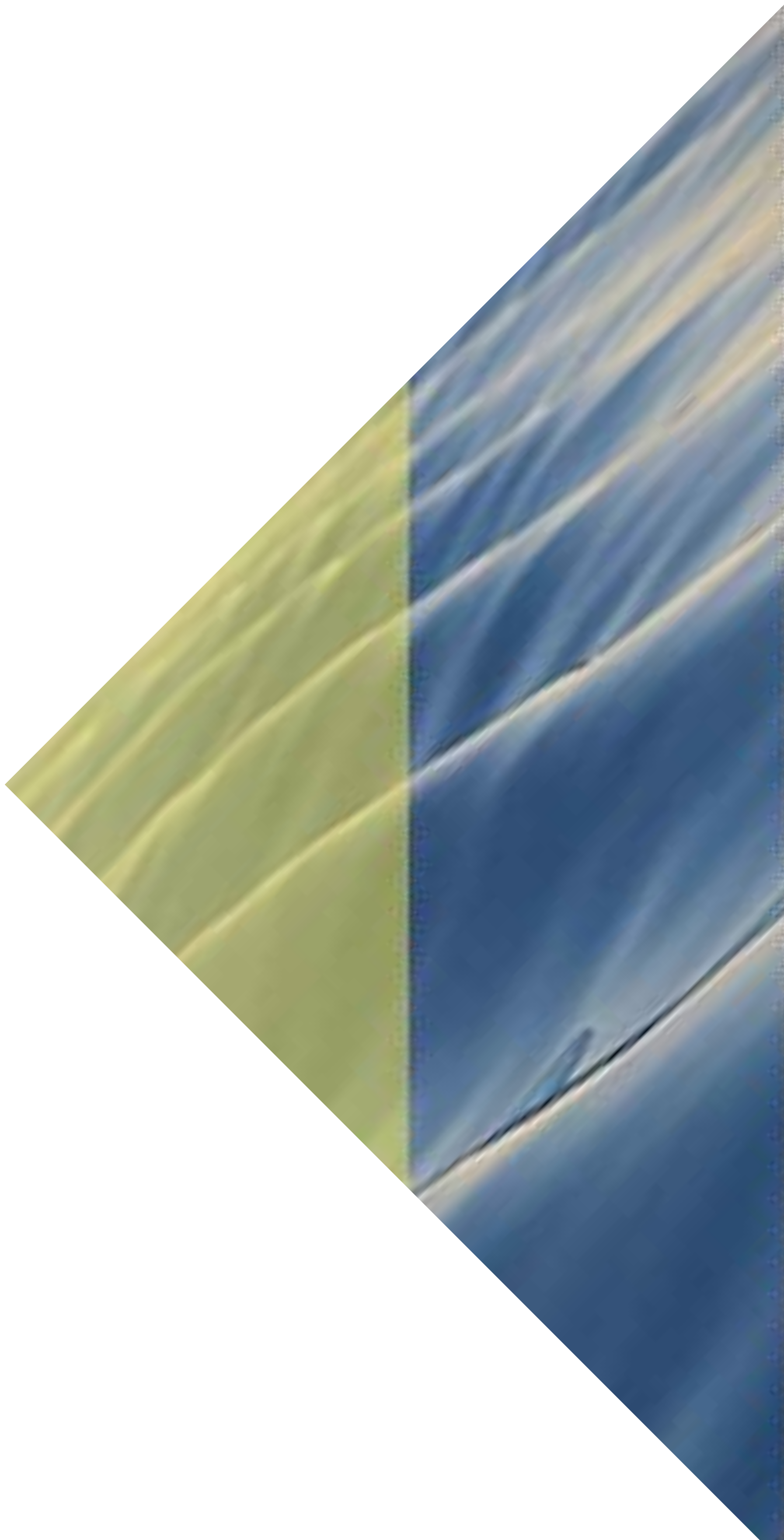
Summary

► We measured the macroscopic quantities of the the typical undular bores arising in four weakly nonlinear water wave models using the DS-fitting method varying the initial condition jump from 0 to ~0.3

► We integrate numerically the water wave undular bores using the Hamiltonian formulation of the water wave problem and we measured the macroscopic quantities for these bores the initial condition jump from 0 to ~0.3

► We compared the leading solitary wave amplitudes and velocities of the four weakly nonlinear models and the free surface Euler equations.

► We found numerically the onset of the Modulation instability for the Whitham-Boussinesq type system and for the water wave equations.



Further Research Problems

DSW en modelo W-B con tensión superficial

$$
\begin{aligned}
u_t &= -\eta_x - u u_x + \alpha \eta_{xxx}, \\
\eta_t &= -\partial_x([\frac{\tanh D}{D}]u) - (\eta u)_x
\end{aligned}
$$

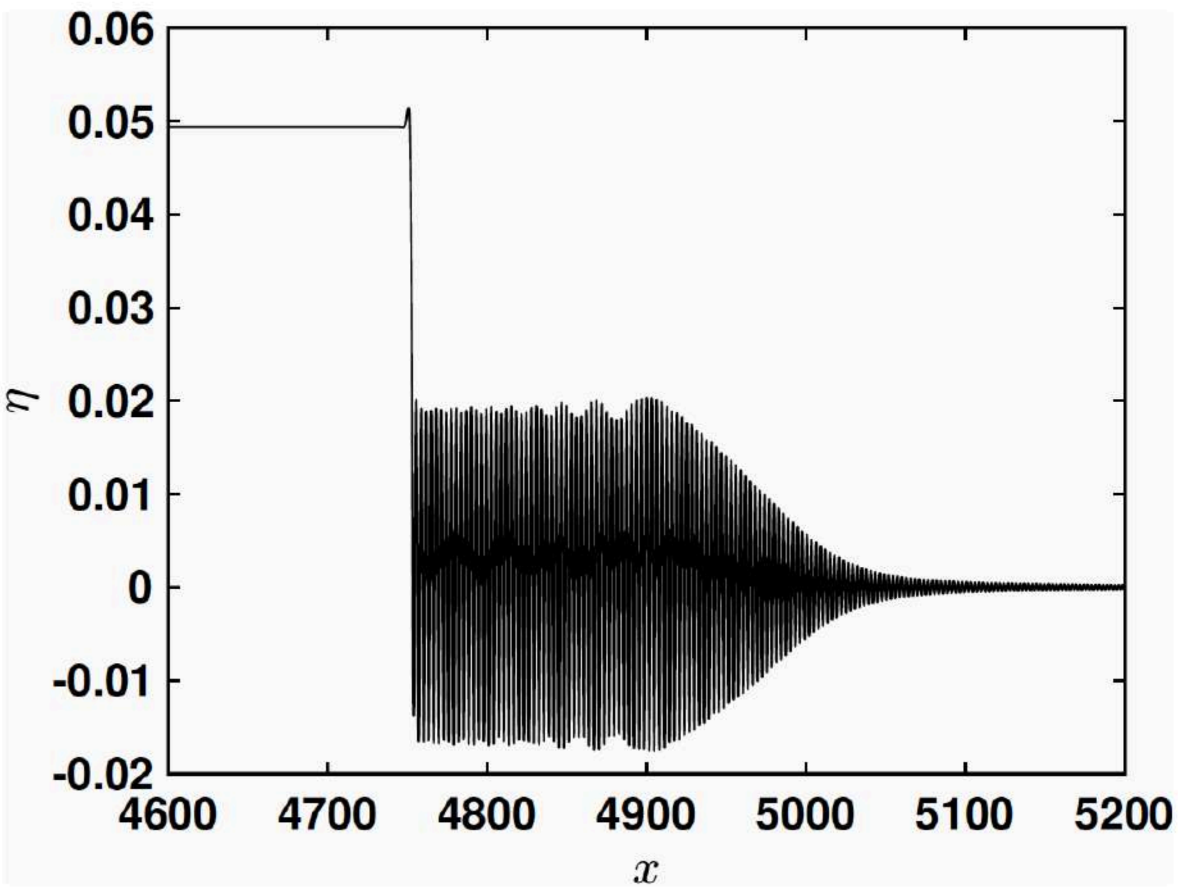


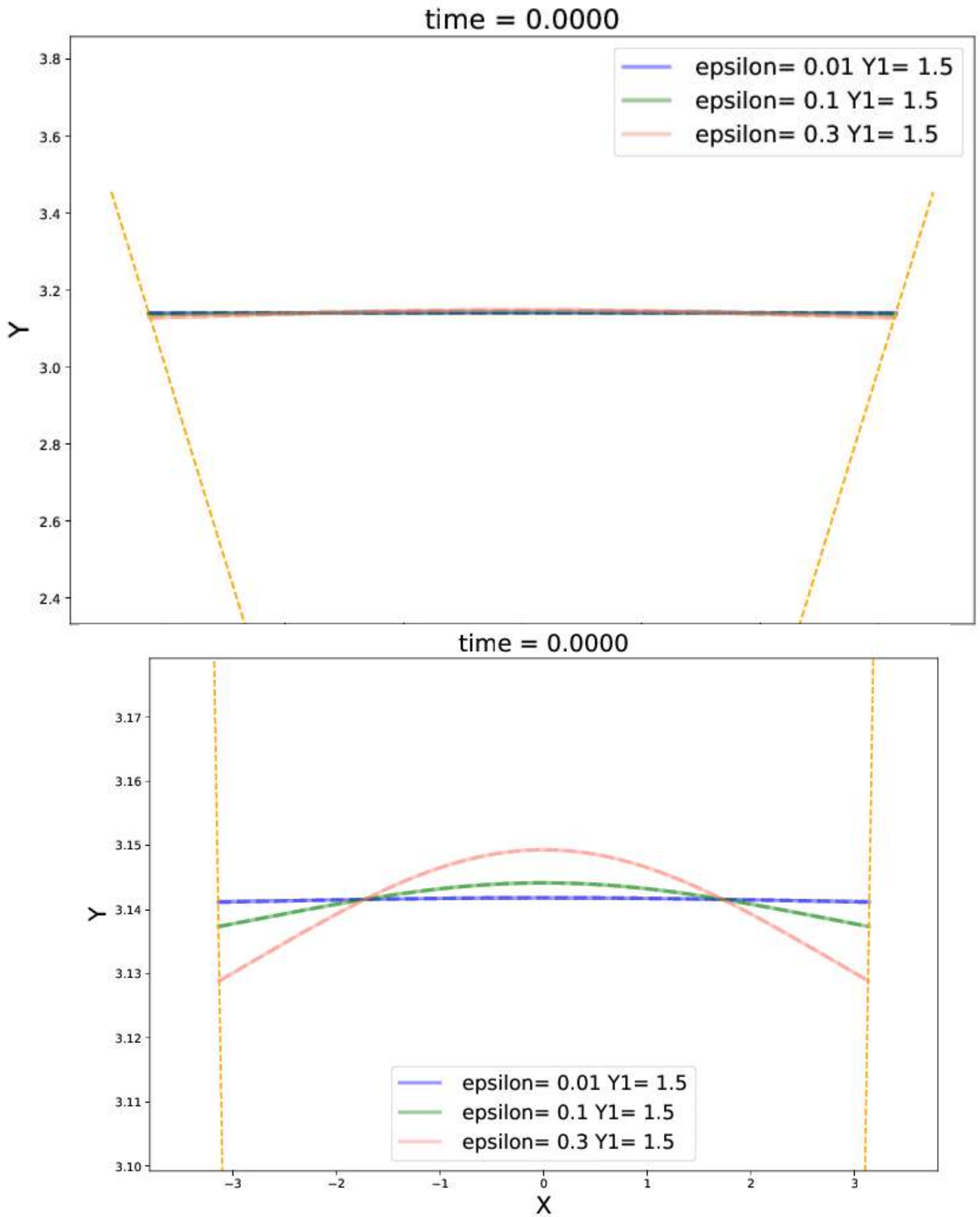
Figure 8: ZOOM IN: Initial jump of 0.1 and $\alpha = 0.3$

Joint work with N. Smyth[1] and T. Marchant[2]

Noel F. Smyth

Tim Marchant

Water wave problem on inclined walls



Joint work with P. Panayotaros

P. Panayotaros