Mini Curso de 6 horas
En beneficio del fortalecimiento al Posgrado Nacional

Institución: CIMAT

Ondas de salto undular. Descripción y métodos de análisis en la actualidad.

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Curso dirigido a : Estudiantes e investigadores del Posgrado en Ciencias con especialidad en Matemáticas Básicas y Aplicadas

Curso abierto a todo público interesado de la comunidad vinculada al Posgrado en Ciencias del CIMAT

Objetivos:

1. Estudiar un tipo genérico de Soluciones Coherentes de Ecuaciones No lineales y Dispersivas. Ondas de Choque Dispersivas (Ondas de Salto Ondular en el contexto de Mecánica de Fluidos)

2. Introducir el Método de ajuste de Choque y aplicarlo a Modelos de ondas en agua débilmente no lineales.

3. Verificar y analizar la posible extensión y exactitud de las medidas macroscópicas para Modelos Completamente No lineales, utilizando el modelo de las ecuaciones de Euler

4. Explorar la Inestabilidad Modulacional de las Ondas de salto Undular que se presentan para tiempos largos de evolución.

Estructura del curso:
Teoría 📚: 3.0 horas,
Trabajo dirigido 📝: 1.5 hora
Discusión 🤓: 1 hora
Descansos ☕: 0.5 horas

Temario:

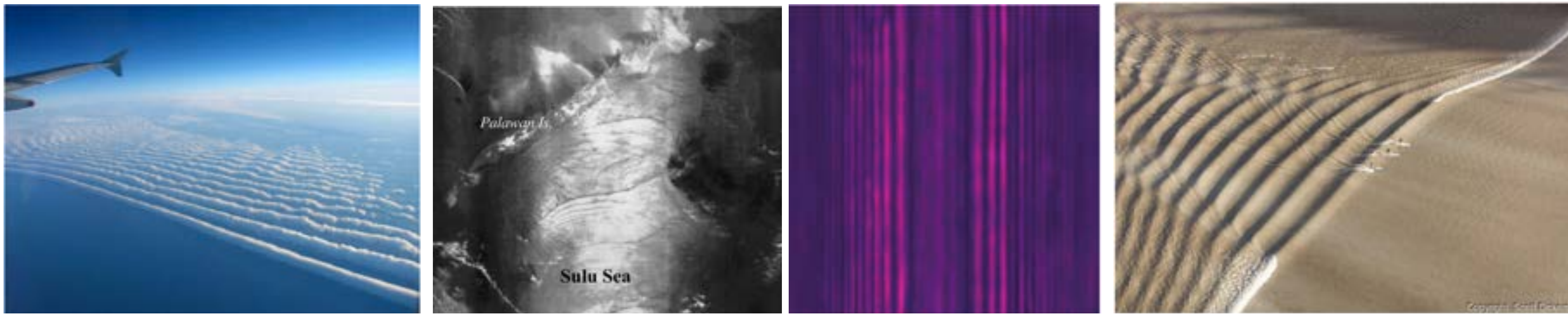
1. **Soluciones de Ecuaciones No lineales y Dispersivas. Ondas de Choque Dispersivas (también llamadas ondas de salto ondular en el contexto de Mecánica de Fluidos)**
   - **Soluciones genéricas coherentes de las Ec. Dif. No lineales y dispersivas**
   - **Anatomía de Ondas de Choque Dispersivas**

2. **Método de ajuste de Choque**
   - **Condiciones Analíticas**
   - **Límite Armónico y Límite de Onda Larga**

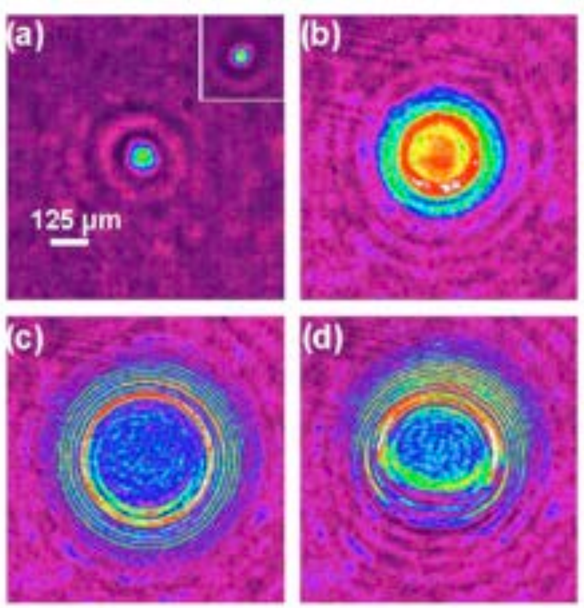
3. **Modelos Débilmente no lineales de ondas en agua**
   - **Sistema A: Boussinesq estándar**
   - **Sistema B: Boussinesq derivado del Hamiltonaino**
   - **Sistema C: Modelo Boussinesq totalmente dispersivo**
   - **Sistema D: Modelo Whitham-Boussinesq**

4. Medidas macroscópicas para las soluciones de salto ondular de los *Modelos Boussinesq*
   - ***Modelo Boussinesq del Hamiltoniano***

1. Ecuaciones No lineales y Dispersivas y Anatomía de Ondas de Choque Dispersivas *(Ondas de Salto Ondular en el contexto de Mecánica de Fluidos)*



En la atmósfera a) Nube de gloria por la mañana. b) Solitones oceánicos internos c) Imagen del conjunto de salida de un cristal foto refractivo desenfocado. d) Salto oscilatorio en un río en Turnagain Arm, Alaska.



Experimental evolution of DSW after 1cm of propagation in an ethanol + iodine cell

Atmospheric gravity waves moving southward off the Texas coast and out over the western Golf of Mexico

¿Qué es una onda de choque dispersiva?

Un DSW es la resolución dispersiva de un frente de onda pronunciado.

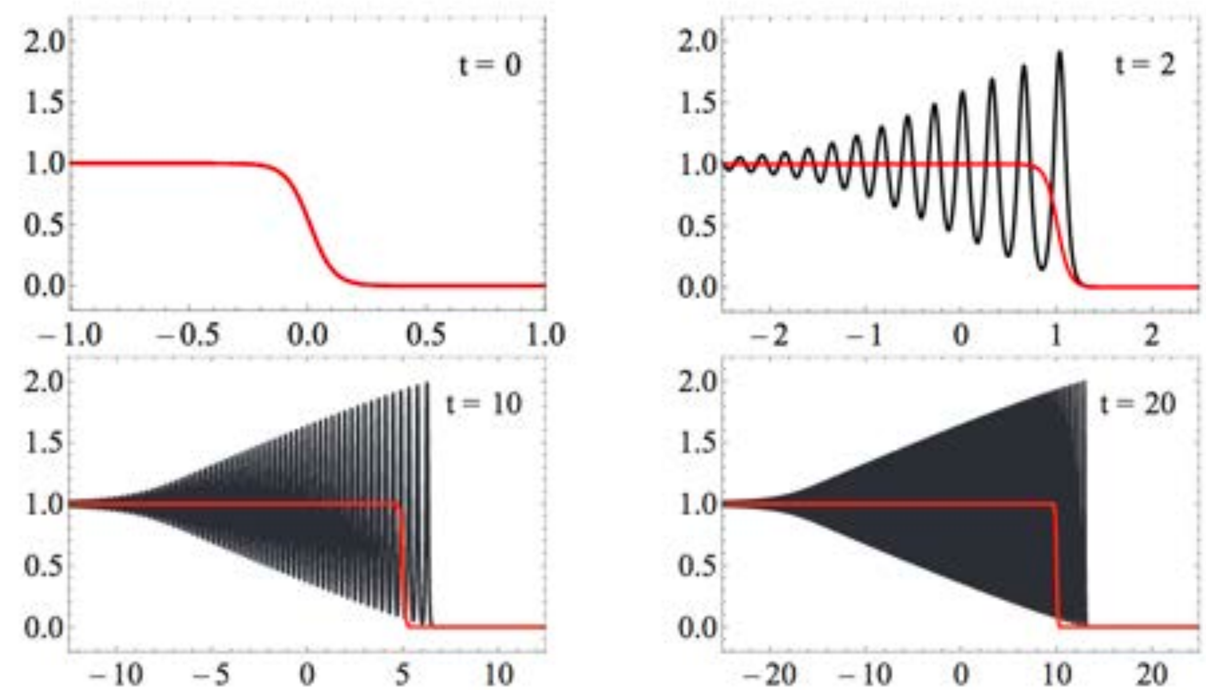
Ecuación de Burgers:
(Modelo más sencillo no lineal y no dispersivo)

$$u_t + uu_x = 0$$

$$u_t + uu_x + u_{xxx} = 0$$

Ecuación KdV:
Modelo más sencillo no lineal y dispersivo

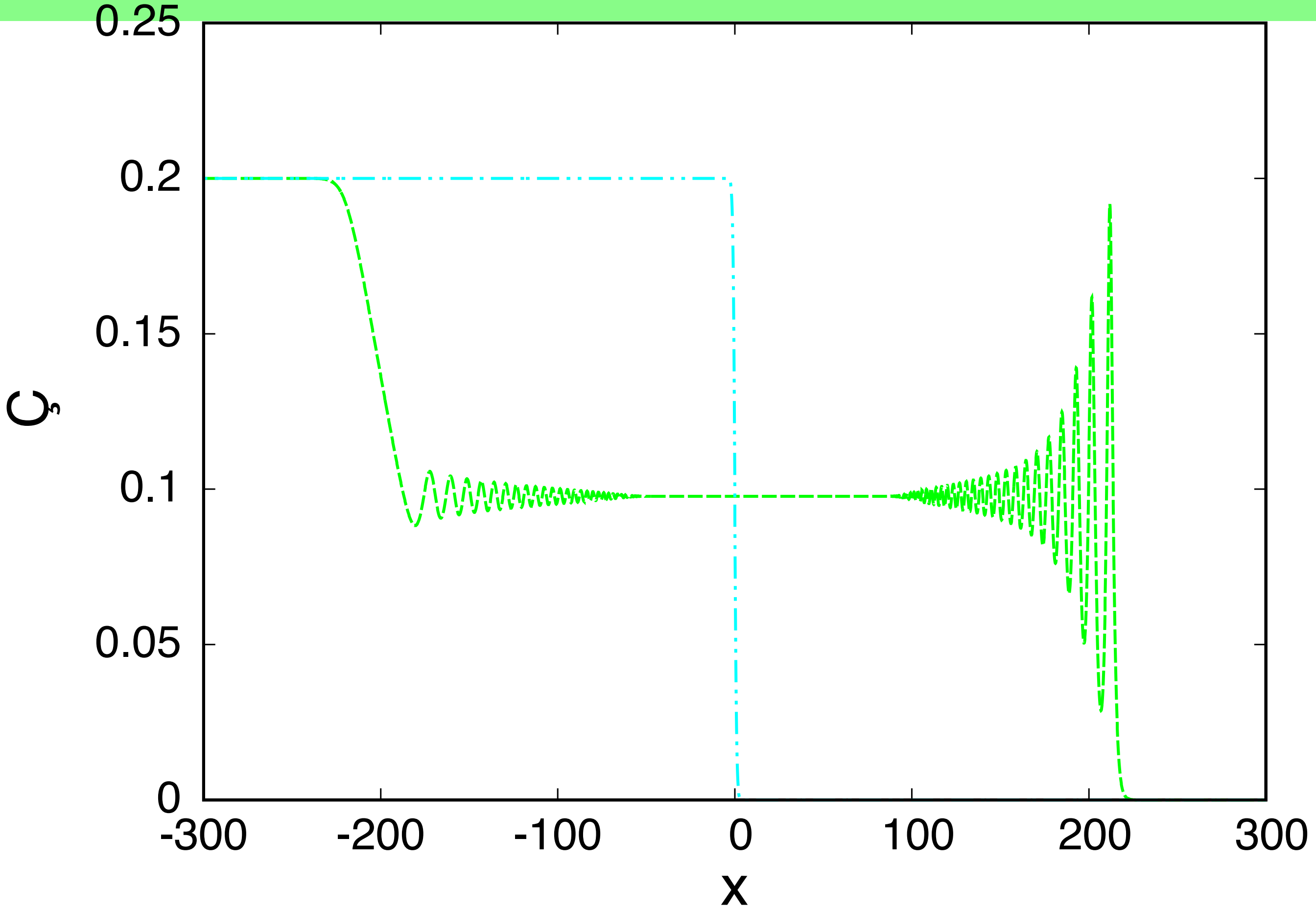
Typical numerically computed solution of the KdV equation is in black and Burgers' equation is in red



El artículo fundamental de Gurevich y Pitaevskii * adaptó el problema de Riemann para considerar el comportamiento a largo plazo de los datos iniciales del paso para la ecuación KdV, una ecuación hiperbólica regularizada de forma dispersa. La solución a este problema es fundamental para la mayor parte de la teoría DSW, nos referimos al problema de KdV Riemann como el problema GP.

[*] A. V. Gurevich, L. P. Pitaevskii, Nonstationary structure of acollisionless shock wave, Sov. Phys. JETP (1974), translation from Russian

Anatomía de un DSW y cantidades macroscópicas



►**R. M. Vargas-Magaña, T. Marchant and N. Smyth** "Numerical and analytical study of undular bores governed by the full water wave equations and bi-directional Whitham-Boussinesq equations" ***Physics of Fluids, 2021 in-press article***

Por medio del método de G. El y colaboradores es posible medir estas cantidades macroscópicas de las ondas de choque dispersivas

- Amplitud de la onda solitaria líder
- Velocidad de la onda solitaria líder
- El número de onda del tren de ondas armónico

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Métodos actuales para determinar analíticamente las **propiedades macroscópicas** de una onda de choque dispersiva **(DSW).**

- ✦ Teoría de modulación
- ✦ **Método de Ajuste de Ondas de choque dispersivas**

► Las ecuaciones de modulación son estrictamente hiperbólicas, no lineales y de **ellas pueden plantearse sistemas reducidos para límites de onda corta** (ondas solitarias) y lineales (límite de tren de ondas armónico).

► La teoría matemática existente para medir analíticamente estas cantidades es válida para cierto régimen físico y no se había probado su validez en sistemas completamente no lineales*

*__R. M. Vargas-Magaña, T. Marchant and N. Smyth__ "Numerical and analytical study of undular bores governed by the full water wave equations and bi-directional Whitham-Boussinesq equations" ***Physics of Fluids, 2021***

2. Método de ajuste de Choque

Un procedimiento de ajuste de DSW determina analíticamente las propiedades macroscópicas de DSW.

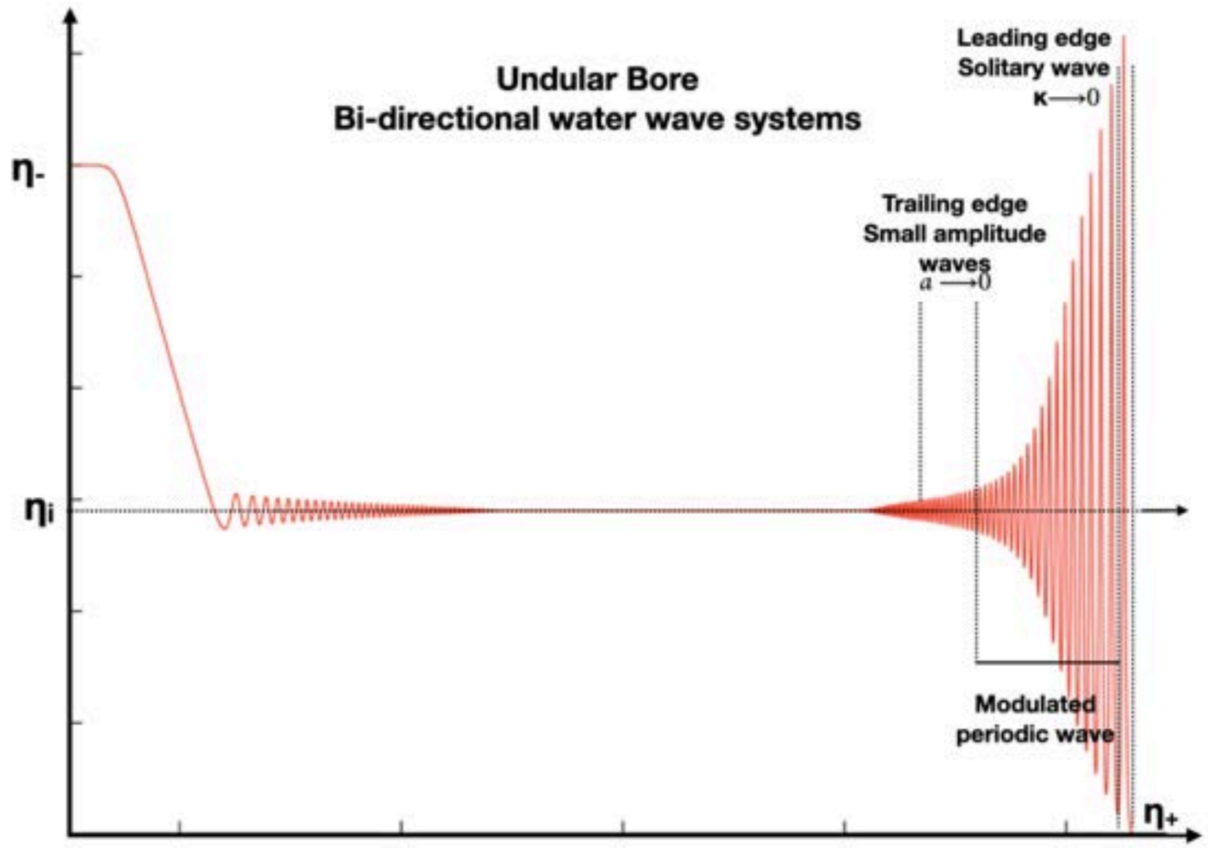
Estas propiedades macroscópicas de DSW incluyen:

- las velocidades de las ondas en los bordes de los DSW
- los parámetros de onda DSW:
  - el número de onda de borde armónico
  - la amplitud del borde del solitón.

La determinación de estos observables representa el ajuste de un DSW a la dinámica a largo plazo de los datos iniciales de Riemann constantes por partes.

2. Método de ajuste de Choque

El método de ajuste DSW propuesto por Gennady El y colaboradores se basa en una propiedad fundamental y genérica: las ecuaciones de modulación de Whitham admiten reducciones exactas a un conjunto de ecuaciones comunes, mucho más simples y analíticamente tratables en los límites de amplitud y número de onda tendientes a cero, que corresponden a los extremos DSW armónico y de soliton líder respectivamente.



2. Método de ajuste de Choque

L as ecuaciones dispersivas no lineales deben satisfacer las siguientes condiciones:

(i) Admite un límite sin dispersión obtenido al introducir las variables lentas.
(ii) La relación de dispersión lineal ω (k) tiene un valor real.
(iii) Posee al menos dos leyes de conservación.
(iv) Admite soluciones periódicas de ondas viajeras, parametrizadas por tres variables independientes.
(v) El sistema de Whitham correspondiente a las dos leyes de conservación promediadas más la ley de conservación del número de onda $kt + \omega x = 0$ es hiperbólico. *La hiperbolicidad no se puede verificar sin resolver directamente el sistema Whitham, por lo que esta condición debe verificarse, por ejemplo, mediante la comparación de resultados teóricos con simulaciones numéricas.*

Método de ajuste de Choque para ecuaciones tipo Whitham

$$u_t + 2uu_x + D_2[u]_x = 0$$

► Las ecuaciones de modulación son estrictamente hiperbólicas, no lineales y tienen límites lineales y solitarios.

► Forma no dispersiva: $\frac{\partial \overline{u}}{\partial t} + 2\overline{u}\frac{\partial \overline{u}}{\partial x} = 0.$

► Relación de dispersión: $\omega(k)$ convexa

$$u_t + 2uu_x + D_2[u]_x = 0$$

- El emparejamiento de la región no dispersiva detrás de la DSW y el borde de salida está determinada por

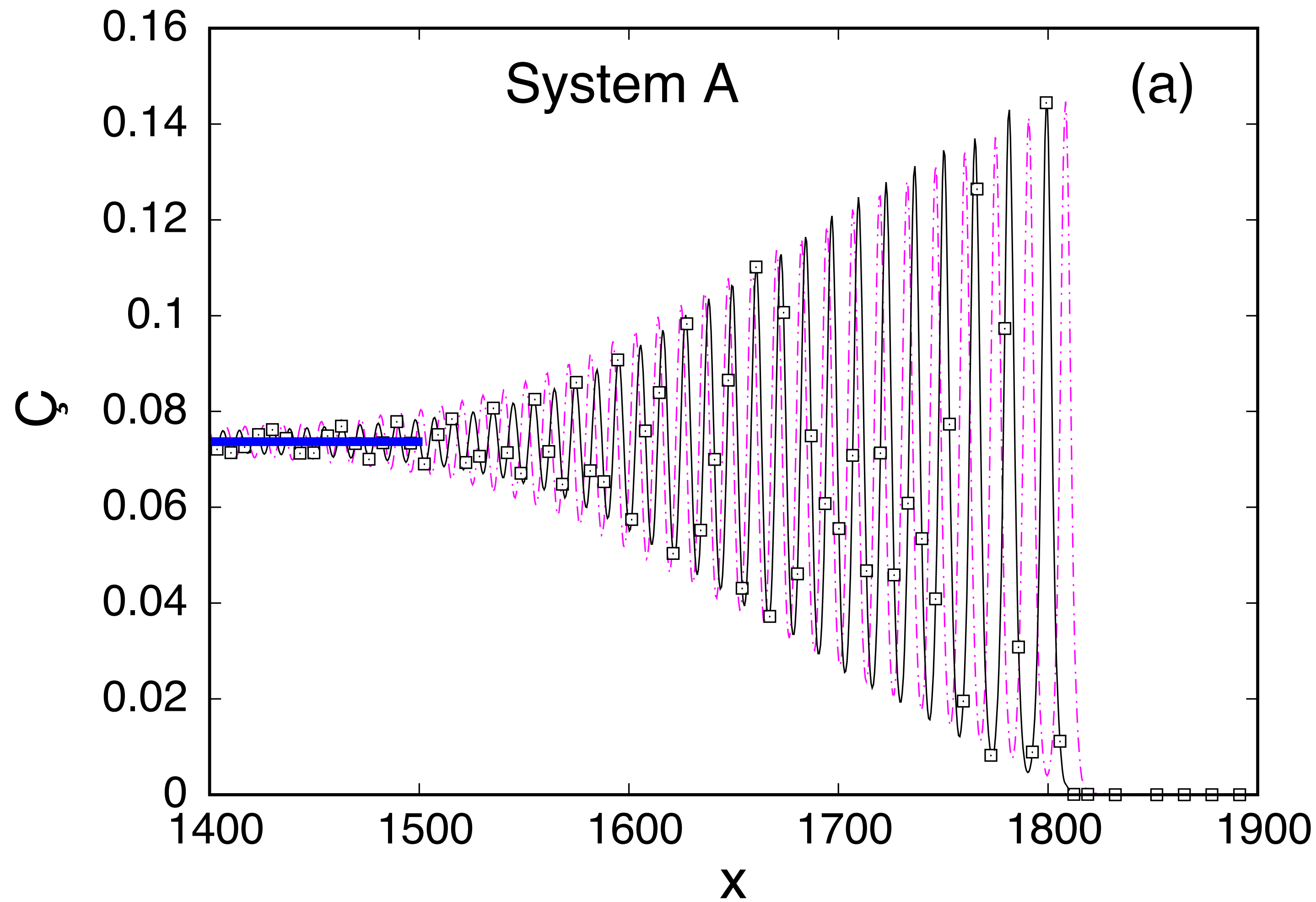
3. Modelos Débilmente no lineales de ondas en agua

- Sistema A: Boussinesq estándar
- Sistema B: Boussinesq derivado del Hamiltoniano
- Sistema C: Modelo Boussinesq totalmente dispersivo
- Sistema D: Modelo Whitham-Boussinesq

Sistema A: Boussinesq Estándar

$$\eta_t = -u_x - (\eta u)_x,$$

$$u_t = -u u_x - \eta_x - \frac{1}{3}\eta_{xxx}$$



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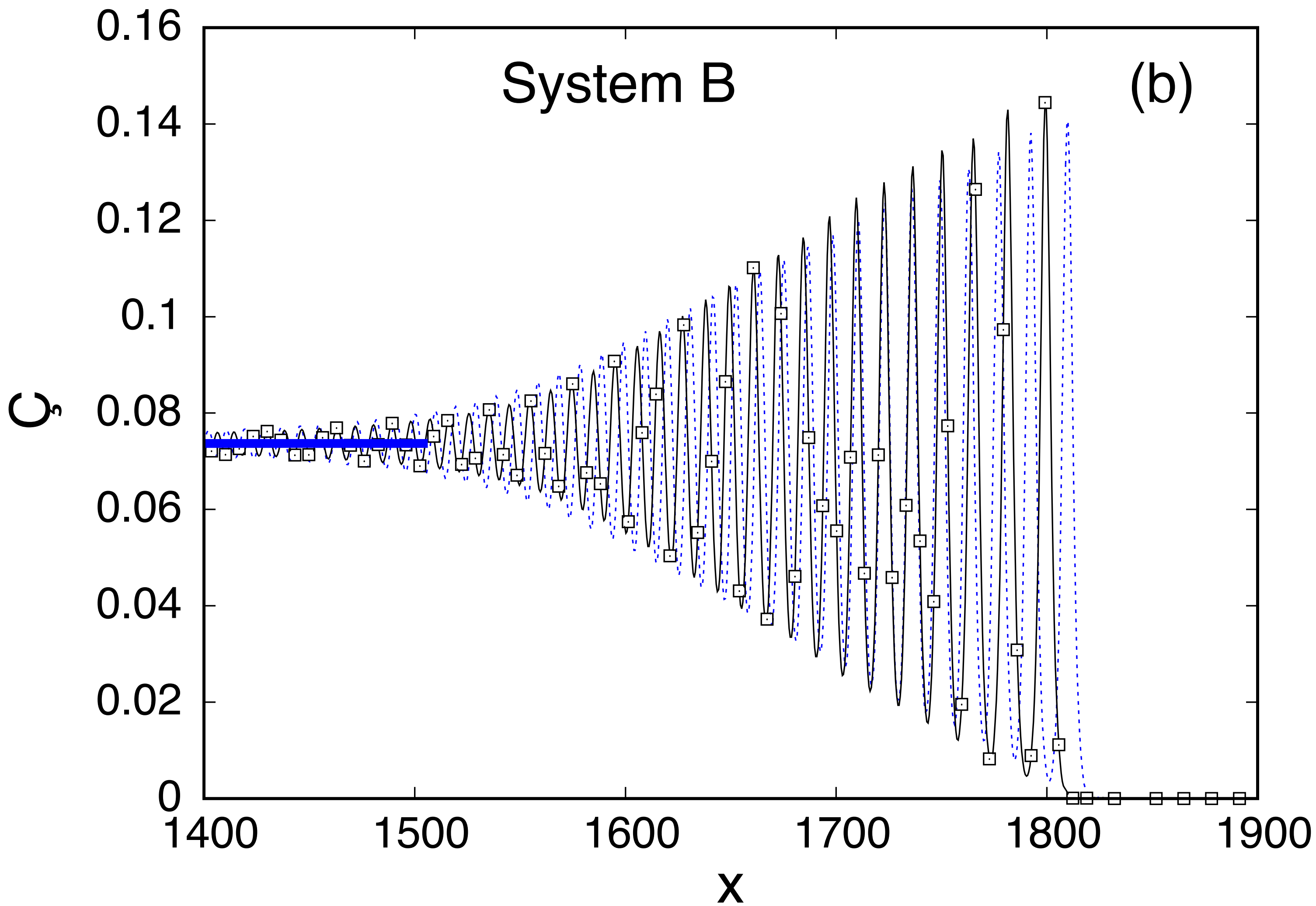
Sistema B: Boussinesq del Hamiltoniano

$$\eta_t = -u_x - \frac{1}{3}u_{xxx} - (\eta u)_x$$

$$u_t = -\eta_x - uu_x.$$



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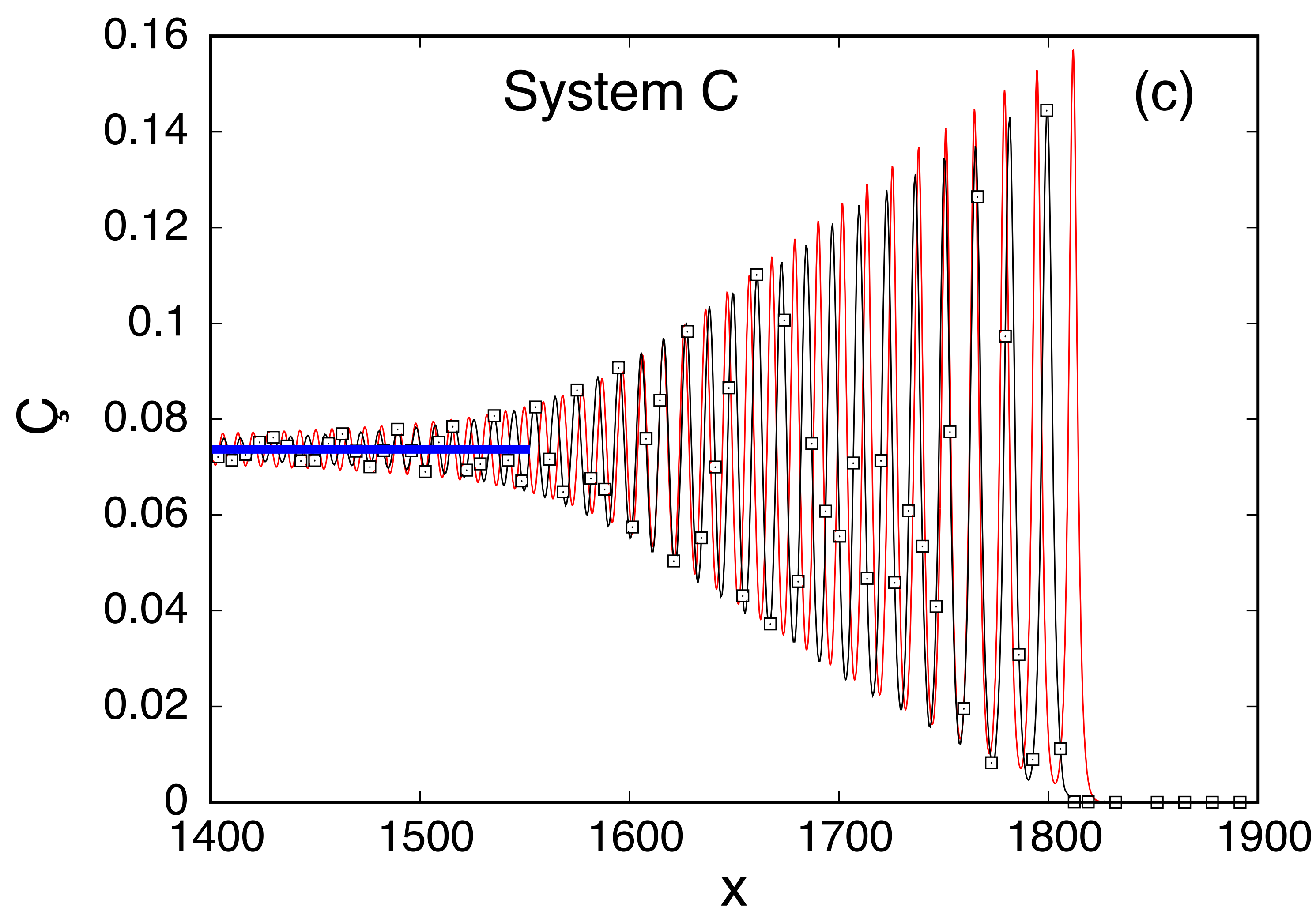
Sistema C:

$$\eta_t = -u_x - (\eta u)_x,$$
$$u_t = -uu_x - \partial_x\left(\left[\frac{\tanh D}{D}\right]\eta\right)$$



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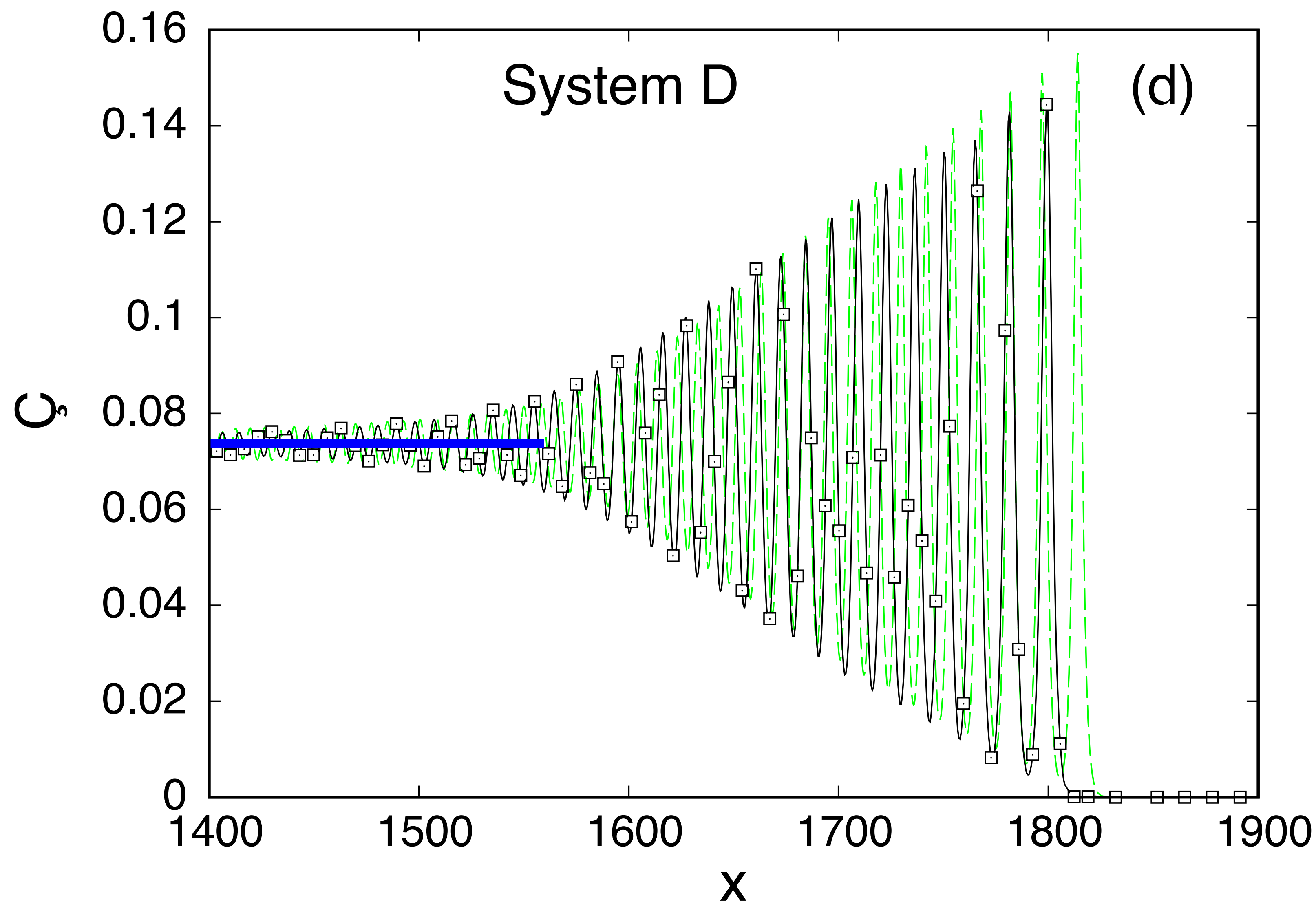
Sistema D: Whitham-Boussineq

$$\eta_t = -\partial_x\left(\left[\frac{\tanh D}{D}\right]u\right) - (\eta u)_x,$$

$$u_t = -\eta_x - uu_x.$$



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All four Boussinesq systems (27)–(30) have the same non-dispersive limit

$$\eta_t + u_x + (\eta u)_x = 0,$$
$$u_t + \eta_x + u u_x = 0.$$

This hyperbolic system can be set in the Riemann invariant form

$$C_+ : u + 2\sqrt{1+\eta} = R_+ \qquad \text{on} \qquad \frac{dx}{dt} = u + \sqrt{1+\eta} = V_+,$$
$$C_- : u - 2\sqrt{1+\eta} = R_- \qquad \text{on} \qquad \frac{dx}{dt} = u - \sqrt{1+\eta} = V_-.$$

The first aim of the analysis is to manipulate equation 3.4 to form a single equation in which the $t-$ and $x-$ derivatives of u and η combine to form derivatives in a particular direction.

Which allows 3.4 to be written as

$$L = \lambda_1[u_t + \frac{\partial x/\partial\sigma}{\partial t/\partial\sigma}u_x] + \lambda_2[\eta_t + \frac{\partial x/\partial\sigma}{\partial t/\partial\sigma}\eta_x] = 0 \tag{3.5}$$

Setting the determinan of the coefficient of λ_1 and λ_2 to zero leads to

$$(\partial x/\partial\sigma - u\partial t/\partial\sigma)(\partial x/\partial\sigma - u\partial t/\partial\sigma) - (\eta\partial t/\partial\sigma - \partial t/\partial\sigma)\partial t/\partial\sigma = 0 \tag{3.6}$$

Then

$$(dx/d\sigma)^2 - 2u(dx/d\sigma)(dt/d\sigma) + (u^2 - \eta)(dt/d\sigma)^2 + (dt/d\sigma)^2 = 0 \tag{3.7}$$

With $(dx/d\sigma)/(dt/d\sigma) = dx/dt$, the characteristic direction (dt, dx) is given by

$$(dx/dt)^2 - 2u(dx/dt) + (u^2 - \eta + 1) = 0 \tag{3.8}$$

This equation has two distinct, real solutions $(\frac{dx}{dt})_-$ and $(\frac{dx}{dt})_+$

This means that the corresponding Riemann invariants and characteristic velocities are:

$$u + 2\sqrt{\eta+1} = R_+ \text{ on } \left(\frac{\partial x}{\partial t}\right)_+ = u + \sqrt{\eta+1} = V_+ \tag{3.9}$$

and

$$u - 2\sqrt{\eta+1} = R_- \text{ on } \left(\frac{\partial x}{\partial t}\right)_- = u - \sqrt{\eta+1} = V_- \tag{3.10}$$

In Gennady's terms $V_+ = (u + \sqrt{\eta+1})$, $V_- = (u - \sqrt{\eta+1})$, $\bar{u} = u$ and $\bar{\eta} = \eta$

Actividad 1: Calcula la relación de dispersión desde un estado estacionario para el sistema A (15 min)

Now we are going to compute the **linear dispersion relation of (0.0.1)** for linear waves propagating on the background $\eta = \bar{\eta}$, $u = \bar{u}$

$$\begin{aligned} \eta_t + u\eta_x + \eta u_x + u_x + \frac{1}{3}u_{xxx} &= 0 \\ u_t + \eta_x + uu_x &= 0 \end{aligned} \tag{2.21}$$

In order to compute the dispersion relation associate to the system above we first "linearize" the governing equations.

In order to linearize the equation we use a *perturbation method*

We start by dividing the variables η and u into two parts:

$$\eta = \bar{\eta} + N \tag{2.22}$$

$$u = \bar{u} + V \tag{2.23}$$

The first part is know as the basic state and we will consider this state constant represented by $\bar{\eta}$ and $\bar{u}$ respectively.

The second part is the *perturbation* and is allowed to vary with time and in the space variable, we denoted this by N and V.

Substituting in equation (0.0.32) and (0.0.33) in equation (0.0.31)

$$\begin{aligned} N_t + (\bar{u} + V)N_x + (\bar{\eta} + N)V_x + V_x + \frac{1}{3}V_{xxx} &= 0 \\ V_t + N_x + (\bar{u} + V)V_x &= 0 \end{aligned} \tag{2.24}$$

$$\begin{aligned} N_t + \bar{u}N_x + VN_x + \bar{\eta}V_x + NV_x + V_x + \frac{1}{3}V_{xxx} &= 0 \\ V_t + N_x + \bar{u}V_x + VV_x &= 0 \end{aligned} \tag{2.25}$$

Since the perturbation quantities are very small, we assume that we can ignore products of perturbation quantities. This further simplifies the equation to

$$\begin{aligned} N_t + \bar{u}N_x + \bar{\eta}V_x + V_x + \frac{1}{3}V_{xxx} &= 0 \\ V_t + N_x + \bar{u}V_x + &= 0 \end{aligned} \tag{2.26}$$

Now we will follow a general method for finding the dispersion relation for the waves by a linearized set of equations as the equations (0.0.37)

Step 1 we will assume that variables η and u have a sinusoidal form

$$N = Ae^{i(kx - \omega t)}$$

$$V = Be^{i(kx - \omega t)}$$

Step 2 to plug the assumed form of the dependent variables into the equations (0.0.37

This yields to the equations:

Step 3 We write equations **Step 2** in matrix form

Step 4 Then we take the determinant of the coefficient matrix and solve for ω.

Actividad 2: Calcula la relación amplitud/velocidad de una onda solitaria para el sistema A (15 min)

– **Standard Boussinesq model**

$$\eta_t = -u_x - (\eta u)_x$$
$$u_t = -uu_x - \eta_x - \frac{1}{3}\eta_{xxx}$$

Considering the moving reference frame $(x - vt)$

$$-v\eta' + u' + u\eta' + \eta u' = 0$$

Integrating

$$-v\eta + u + u\eta = 0$$

Then we get

$$u = \frac{v\eta}{1+\eta}$$

Using the second equation we obtain

$$-vu' + uu' + g\eta' + \frac{1}{3}\eta''' = 0$$

Integrating

$$-vu + \frac{1}{2}u^2 + g\eta + \frac{1}{3}\eta'' = 0$$

then using both equations and subtituting 25 in 83 we obtain

$$A = \frac{-v^2\eta}{2} - \frac{v^2}{2(1+\eta)} + g\frac{\eta^2}{2} + \frac{1}{6}\eta'^2$$

Assuming $A = -\frac{1}{2}v^2$
Then

$$-\frac{1}{2}v^2 = \frac{-v^2\eta}{2} - \frac{v^2}{2(1+\eta)} + g\frac{\eta^2}{2} + \frac{1}{6}\eta'^2$$

Assuming $\eta' = 0$ and $g = 1$

$$\frac{1}{2}v^2\frac{a^2}{(1+a)} - g\frac{a^2}{2} = 0$$

$$v = \sqrt{1+a} \sim 1 + \frac{a}{2}$$

Actividad 3: Aplica el método de onda de Choque dispersiva para el sistema A (30 min)

Standard Boussinesq Model

$$\omega^{SB}(\bar\rho,k) = 2(\sqrt{\bar\rho}-1)k + k\sqrt{\bar\rho(1-\frac{k^2}{3})}$$

- $$\frac{\partial\omega^{SB}(\bar\rho,k)}{\partial\bar\rho} = \frac{2k}{2\sqrt{\bar\rho}} + \frac{k(1-\frac{k^2}{3})}{2\sqrt{\bar\rho(1-\frac{k^2}{3})}} = \frac{k}{\sqrt{\bar\rho}}(1+\frac{(1-\frac{k^2}{3})}{2\sqrt{(1-\frac{k^2}{3})}})$$

- $$\begin{aligned}\frac{\partial\omega^{SB}(\bar\rho,k)}{\partial k} &= 2(\sqrt{\bar\rho}-1) + \sqrt{\bar\rho(1-\frac{k^2}{3})} + \frac{k(\frac{-2\bar\rho k}{3})}{2\sqrt{\bar\rho(1-\frac{k^2}{3})}} \\ &= 2(\sqrt{\bar\rho}-1) + \sqrt{\bar\rho(1-\frac{k^2}{3})} - \frac{\bar\rho k^2}{3\sqrt{\bar\rho(1-\frac{k^2}{3})}}\end{aligned}$$

-

Then EDO associated is:

$$\begin{aligned}\frac{dk}{d\bar\rho} &= \frac{\omega_\rho^{SB}(\bar\rho,k)}{V(\rho)-\omega_k^{SB}(\bar\rho,k)} = \frac{\frac{k}{\sqrt{\bar\rho}}(1+\frac{(1-\frac{k^2}{3})}{2\sqrt{(1-\frac{k^2}{3})}})}{3\sqrt{\bar\rho}-2-(2(\sqrt{\bar\rho}-1)+\sqrt{\bar\rho(1-\frac{k^2}{3})}-\frac{\bar\rho k^2}{3\sqrt{\bar\rho(1-\frac{k^2}{3})}})} \\
&= \frac{\frac{k}{\sqrt{\bar\rho}}(1+\frac{(1-\frac{k^2}{3})}{2\sqrt{(1-\frac{k^2}{3})}})}{\sqrt{\bar\rho}-\sqrt{\bar\rho(1-\frac{k^2}{3})}+\frac{\bar\rho k^2}{3\sqrt{\bar\rho(1-\frac{k^2}{3})}})} \\
&= \frac{\frac{k}{\sqrt{\bar\rho}}(1+\frac{(1-\frac{k^2}{3})}{2\sqrt{(1-\frac{k^2}{3})}})}{\sqrt{\bar\rho}(1-\sqrt{1-\frac{k^2}{3}})+\frac{k^2}{3\sqrt{(1-\frac{k^2}{3})}})} = \frac{k}{\bar\rho}\frac{(1+\frac{(1-\frac{k^2}{3})}{2\sqrt{(1-\frac{k^2}{3})}})}{(1-\sqrt{1-\frac{k^2}{3}})+\frac{k^2}{3\sqrt{(1-\frac{k^2}{3})}})} \\
&= \frac{k}{\bar\rho}\frac{\frac{2\sqrt{1-\frac{k^2}{3}}+(1-\frac{k^2}{3})}{2\sqrt{1-\frac{k^2}{3}}}}{(1-\sqrt{1-\frac{k^2}{3}})+\frac{k^2}{3\sqrt{(1-\frac{k^2}{3})}})} = \frac{k}{\bar\rho}\frac{2\sqrt{1-\frac{k^2}{3}}+(1-\frac{k^2}{3})}{(2\sqrt{1-\frac{k^2}{3}})(1-\sqrt{1-\frac{k^2}{3}})+\frac{k^2}{3\sqrt{(1-\frac{k^2}{3})}})} \\
&= \frac{k}{\bar\rho}\frac{2\sqrt{1-\frac{k^2}{3}}+(1-\frac{k^2}{3})}{(2\sqrt{1-\frac{k^2}{3}}-2(1-\frac{k^2}{3})+\frac{2k^2}{3})}\end{aligned}$$

Let's consider the change of variable described by $\alpha = \frac{\omega^{SB}}{\sqrt{\bar\rho}k} = \frac{k\sqrt{\bar\rho(1-\frac{k^2}{3})}}{\sqrt{\bar\rho}k} = \sqrt{1-\frac{k^2}{3}}$

then $\alpha^2 = (1-\frac{k^2}{3})$, then $3(1-\alpha^2) = k^2$ that lead us to

$$2\alpha\frac{d\alpha}{d\bar\rho} = -\frac{2}{3}k\frac{dk}{d\bar\rho}$$

And substituting α in equation (7.46) lead us to

$$\frac{dk}{d\bar\rho} = \frac{k}{\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+\frac{2k^2}{3})}$$

Then equating equation 7.58 in 7.59 we get

$$\frac{-3\alpha}{k}\frac{d\alpha}{d\bar\rho} = \frac{k}{\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+\frac{2k^2}{3})}$$

$$\begin{aligned}\frac{d\alpha}{d\bar\rho} &= \frac{k^2}{-3\alpha\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+\frac{2k^2}{3})} = \frac{3(1-\alpha^2)}{-3\alpha\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+\frac{2(3(1-\alpha^2))}{3})} \\
&= \frac{(\alpha^2-1)}{\alpha\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+2-2\alpha^2)} = \frac{(\alpha^2-1)}{\alpha\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha-4\alpha^2+2)} \\
&= \frac{(\alpha+1)(\alpha-1)}{\alpha\bar\rho}\frac{2\alpha+\alpha^2}{(2\alpha+1)2(1-\alpha)} = \frac{-(\alpha+1)\alpha(2+\alpha)}{\alpha\bar\rho(2\alpha+1)2} = \frac{-(\alpha+1)(2+\alpha)}{2\bar\rho(2\alpha+1)}\end{aligned}$$

Actividad 4: Encuentra la solución de la EDO's para el límite de onda corta

(30 min)

$$
\begin{aligned}
\frac{d\alpha}{d\bar{\rho}} &= \frac{k^2}{-3\alpha\bar{\rho}}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+\frac{2k^2}{3})} = \frac{3(1-\alpha^2)}{-3\alpha\bar{\rho}}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+\frac{2(3(1-\alpha^2))}{3})}\\
&= \frac{(\alpha^2-1)}{\alpha\bar{\rho}}\frac{2\alpha+\alpha^2}{(2\alpha-2\alpha^2+2-2\alpha^2)} = \frac{(\alpha^2-1)}{\alpha\bar{\rho}}\frac{2\alpha+\alpha^2}{(2\alpha-4\alpha^2+2)}\\
&= \frac{(\alpha+1)(\alpha-1)}{\alpha\bar{\rho}}\frac{2\alpha+\alpha^2}{(2\alpha+1)2(1-\alpha)} = \frac{-(\alpha+1)\alpha(2+\alpha)}{\alpha\bar{\rho}(2\alpha+1)2} = \frac{-(\alpha+1)(2+\alpha)}{2\bar{\rho}(2\alpha+1)}
\end{aligned}
$$

and we get a variable separable equation.

$$\frac{(2\alpha+1)d\alpha}{(\alpha+1)(2+\alpha)} = -\frac{d\bar{\rho}}{2\bar{\rho}}$$

Integrating this equation we wet

$$\int\frac{2\alpha+1+1-1d\alpha}{(\alpha+1)(2+\alpha)} = -\frac{1}{2}\int\frac{d\bar{\rho}}{\bar{\rho}}$$

This is

$$\int\frac{2d\alpha}{(2+\alpha)} - \int\frac{d\alpha}{(\alpha+1)(2+\alpha)} = -\frac{1}{2}\int\frac{d\bar{\rho}}{\bar{\rho}}$$

This is

$$\int\frac{2d\alpha}{(2+\alpha)} - \int\frac{d\alpha}{(\alpha+1)} + \int\frac{d\alpha}{(2+\alpha)} = -\frac{1}{2}\int\frac{d\bar{\rho}}{\bar{\rho}}$$

Then

$$2\ln(2+\alpha) - \ln(\alpha+1) + \ln(2+\alpha) = \ln(\bar{\rho})^{-\frac{1}{2}} + C \tag{5.19}$$

This is

$$\ln\frac{(2+\alpha)^3}{(\alpha+1)} = -\ln(\bar{\rho})^{\frac{1}{2}} + C \tag{5.20}$$

This is

$$\frac{(2+\alpha)^3}{(\alpha+1)} = \frac{A}{\sqrt{\bar{\rho}}} \tag{5.21}$$

The solution of the differential equation above with the boundary condition At $k=0$ $\alpha(0)=1$ and $\eta_+ = 0$ then $\bar{\rho}_+ = 1$, ensuring consistency with the leading, solitary wave edge of the DSW, is

$$\frac{(2+1)^3}{(1+1)} = \frac{A}{\sqrt{1}} \tag{5.22}$$

$A = (27/2)$

Then

$$\frac{(2+\alpha)^3}{(\alpha+1)} = \frac{27}{2\sqrt{\bar{\rho}}} \tag{5.23}$$

$$\frac{2(2+\alpha)^3}{27(\alpha+1)} = \frac{1}{\sqrt{\bar{\rho}}} \tag{5.24}$$

$$\frac{27(\alpha+1)}{2(2+\alpha)^3} = \sqrt{\bar{\rho}} \tag{5.25}$$

DISCUSIÓN GRUPAL (10 min)
Aplicabilidad del Método

	Matching Equations	Leading soliton wave Amplitude/velocity relation
System A	Explicit solutions	✓
System B	Explicit solutions	✓
System C	Explicit ODE's equations/ numerical solutions	✗
System D	Explicit ODE's equations/ numerical solutions	✗

¿cuáles son los inconvenientes para aplicar el método de ajuste de choque para un sistema completamente no lineal como las Ecuaciones de superficie de Euler?

Sistema A: Boussinesq Estándar

$$\omega^{System A}(\bar{\eta}, k) = \bar{u}k + k\sqrt{(1+\bar{\eta})\left(1-\frac{1}{3}k^2\right)}$$

$$\frac{dk}{d\bar{\eta}} = \frac{\frac{\partial \omega^{System A}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \omega^{System A}}{\partial k}} = \frac{k}{2(1+\bar{\eta})} \frac{2\sqrt{1-\frac{1}{3}k^2} + 1 - \frac{1}{3}k^2}{\sqrt{1-\frac{1}{3}k^2} - 1 + \frac{2}{3}k^2}$$

3.1 Dispersion relation from a background state $(\bar{\eta}, \bar{u})$ associated with the weakly non-linear models

- Dispersion relation for the Standard Boussinesq model

$$
\begin{aligned}
\omega^{SB}(\bar{\rho}, k) &= 2(\sqrt{\bar{\rho}} - 1)k + k\sqrt{\bar{\rho}(1 - \frac{k^2}{3})} && (3.13) \\
& && (3.14)
\end{aligned}
$$

where $\bar{\rho} = \bar{\eta} + 1$ And the conjugate dispersion relation is:

$$
\begin{aligned}
\tilde{\omega}^{SB}(\bar{\rho}, \tilde{k}) &= -i\omega_{+}^{SB}(\bar{\rho}, i\tilde{k}) && (3.15) \\
&= -i[2(\sqrt{\bar{\rho}} - 1)(i\tilde{k}) + i\tilde{k}\sqrt{\bar{\rho}(1 - \frac{(i\tilde{k})^2}{3})}] && (3.16) \\
&= 2(\sqrt{\bar{\rho}} - 1)\tilde{k} + \tilde{k}\sqrt{\bar{\rho}(1 + \frac{\tilde{k}^2}{3})} && (3.17)
\end{aligned}
$$

$$
\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\frac{\partial \tilde{\omega}^{SystemA}}{\partial \bar{\eta}}}{V_{+} - \frac{\partial \tilde{\omega}^{SystemA}}{\partial \tilde{k}}} = \frac{\tilde{k}}{2(1+\bar{\eta})} \frac{2\sqrt{1 + \frac{1}{3}\tilde{k}^2} + 1 + \frac{1}{3}\tilde{k}^2}{\sqrt{1 + \frac{1}{3}\tilde{k}^2} - 1 - \frac{2}{3}\tilde{k}^2}
$$

Sistema B: Boussinesq del Hamiltoniano

$$\omega^{System\,B} = \bar{u}k + k\left[1 + \bar{\eta} - \frac{1}{3}k^2\right]^{1/2}$$

$$\frac{dk}{d\bar{\eta}} = \frac{\frac{\partial \omega^{System\,B}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \omega^{System\,B}}{\partial k}} = \frac{k}{\bar{\rho}} \frac{(2\sqrt{1 - \frac{k^2}{3\bar{\rho}}} + 1)}{2\sqrt{1 - \frac{k^2}{3\bar{\rho}}} - 2(1 - \frac{k^2}{3\bar{\rho}}) + \frac{2k^2}{3\bar{\rho}}}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\frac{\partial \tilde{\omega}^{System\,B}}{\partial \bar{\eta}}}{V_+ - \frac{\partial \tilde{\omega}^{System\,B}}{\partial \tilde{k}}} = \frac{\tilde{k}}{\bar{\rho}} \frac{(2\sqrt{1 + \frac{\tilde{k}^2}{3\bar{\rho}}} + 1)}{2\sqrt{1 + \frac{\tilde{k}^2}{3\bar{\rho}}} - 2(1 + \frac{\tilde{k}^2}{3\bar{\rho}}) - \frac{2\tilde{k}^2}{3\bar{\rho}}}$$

Sistema C:

$$\omega^{System\,C} = \bar{u}k + \sqrt{(1+\bar{\eta})k\tanh k}.$$

$$\frac{dk}{d\bar{\eta}} = \frac{\sqrt{k\tanh k}}{1+\bar{\eta}}\frac{k+\frac{1}{2}\sqrt{k\tanh k}}{\sqrt{k\tanh k}-\frac{1}{2}\tanh k-\frac{1}{2}k\,\mathrm{sech}^2 k}$$

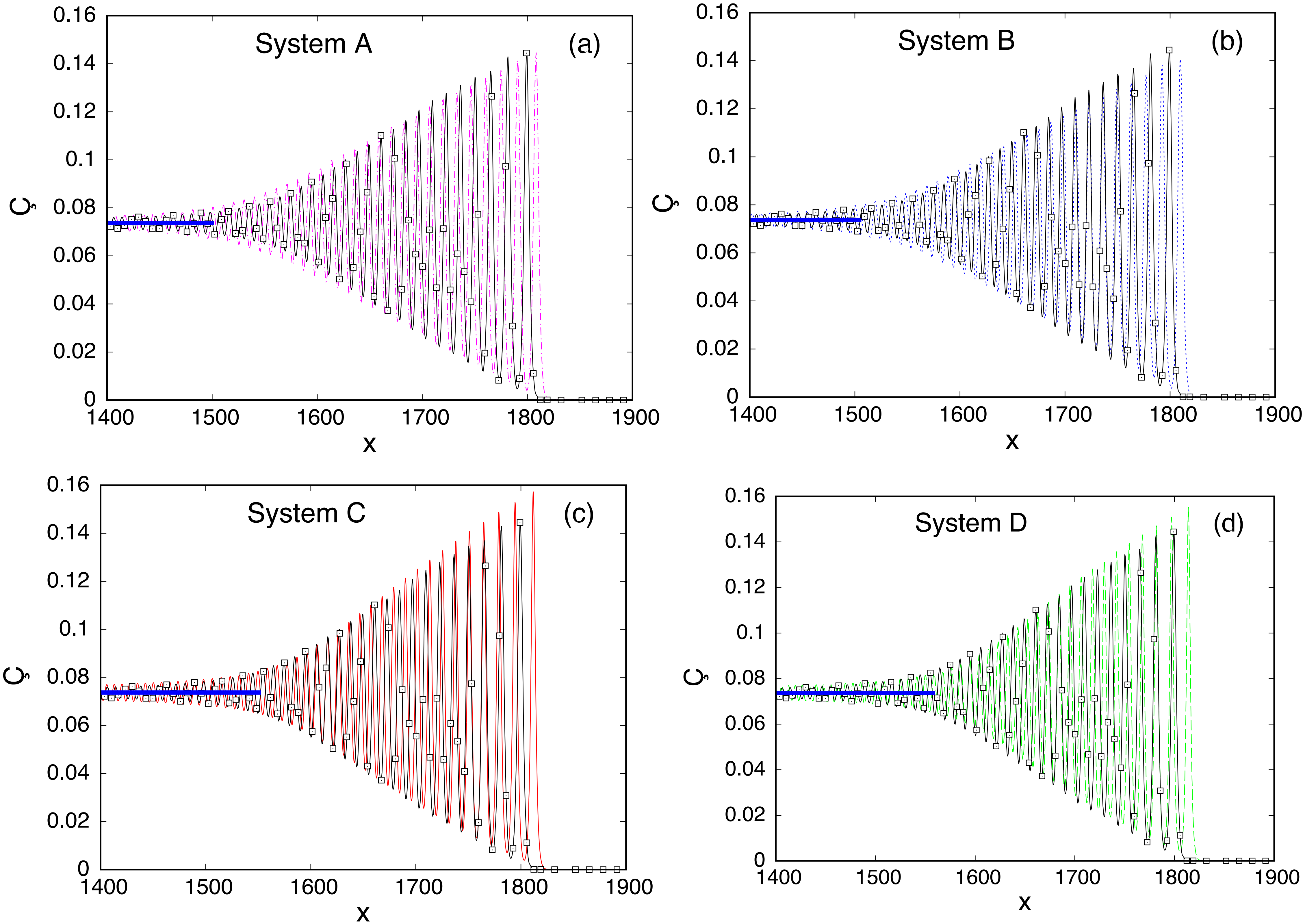
$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\sqrt{\tilde{k}\tan\tilde{k}}}{1+\bar{\eta}}\frac{\tilde{k}+\frac{1}{2}\sqrt{\tilde{k}\tan\tilde{k}}}{\sqrt{\tilde{k}\tan\tilde{k}}-\frac{1}{2}\tan\tilde{k}-\frac{1}{2}\tilde{k}\sec^2\tilde{k}}$$

Sistema D: Whitham-Boussineq

$$\omega^{System D} = \bar{u}k + k\left[\frac{\tanh k}{k} + \bar{\eta}\right]^{1/2}$$

$$\frac{dk}{d\bar{\eta}} = \frac{k}{\sqrt{1+\bar{\eta}}}\frac{\sqrt{\frac{\tanh k}{k}+\bar{\eta}}+\frac{1}{2}\sqrt{1+\bar{\eta}}}{\sqrt{1+\bar{\eta}}\sqrt{\frac{\tanh k}{k}+\bar{\eta}}-\bar{\eta}-\frac{1}{2}\frac{\tanh k}{k}-\frac{1}{2}\operatorname{sech}^2 k}$$

$$\frac{d\tilde{k}}{d\bar{\eta}} = \frac{\tilde{k}}{\sqrt{1+\bar{\eta}}}\frac{\sqrt{\frac{\tan\tilde{k}}{\tilde{k}}+\bar{\eta}}+\frac{1}{2}\sqrt{1+\bar{\eta}}}{\sqrt{1+\bar{\eta}}\sqrt{\frac{\tan\tilde{k}}{\tilde{k}}+\bar{\eta}}-\bar{\eta}-\frac{1}{2}\frac{\tan\tilde{k}}{\tilde{k}}-\frac{1}{2}\sec^2\tilde{k}}$$



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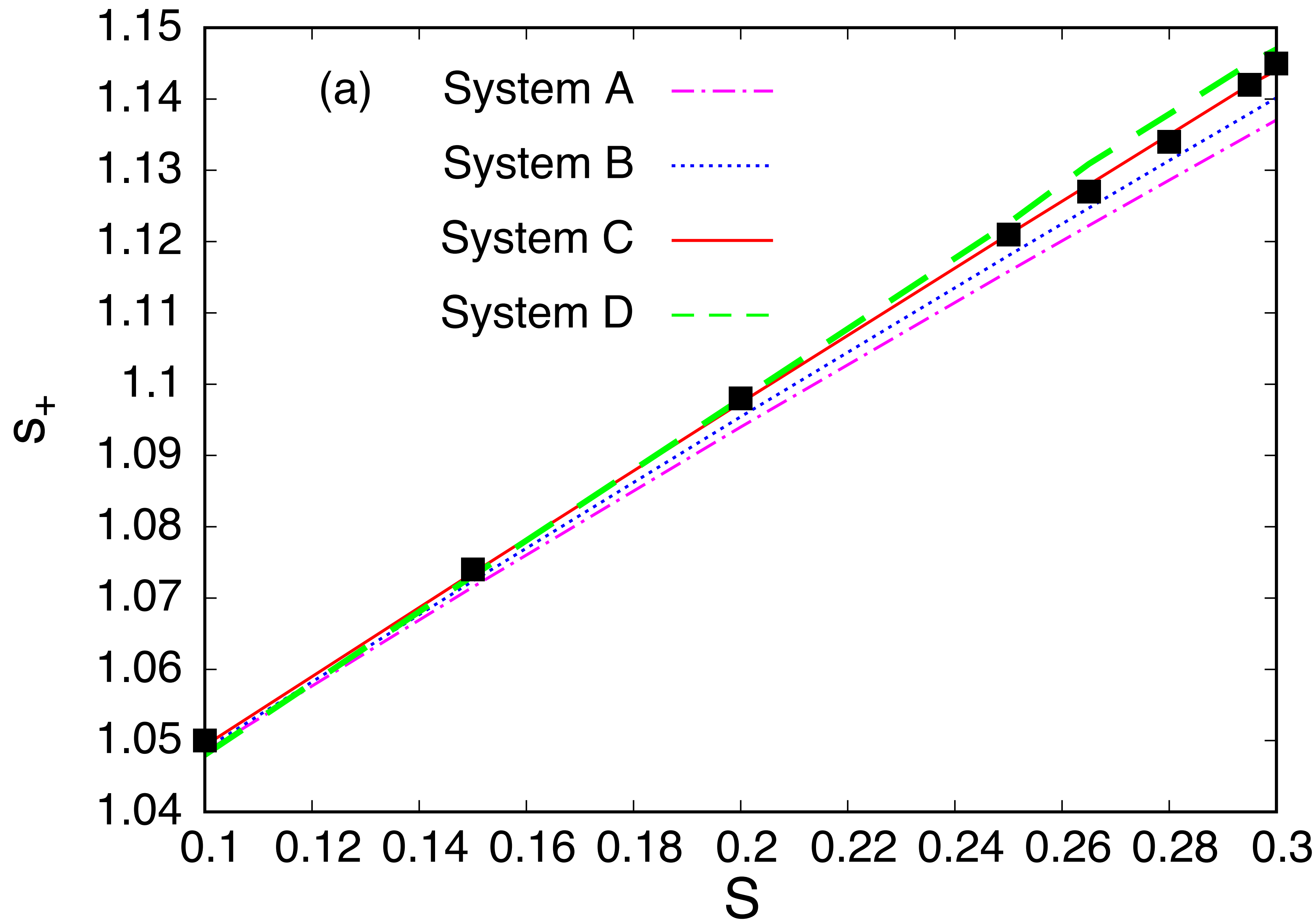
3. Medidas macroscópicas para las soluciones de salto ondular de los *Modelos Boussinesq*



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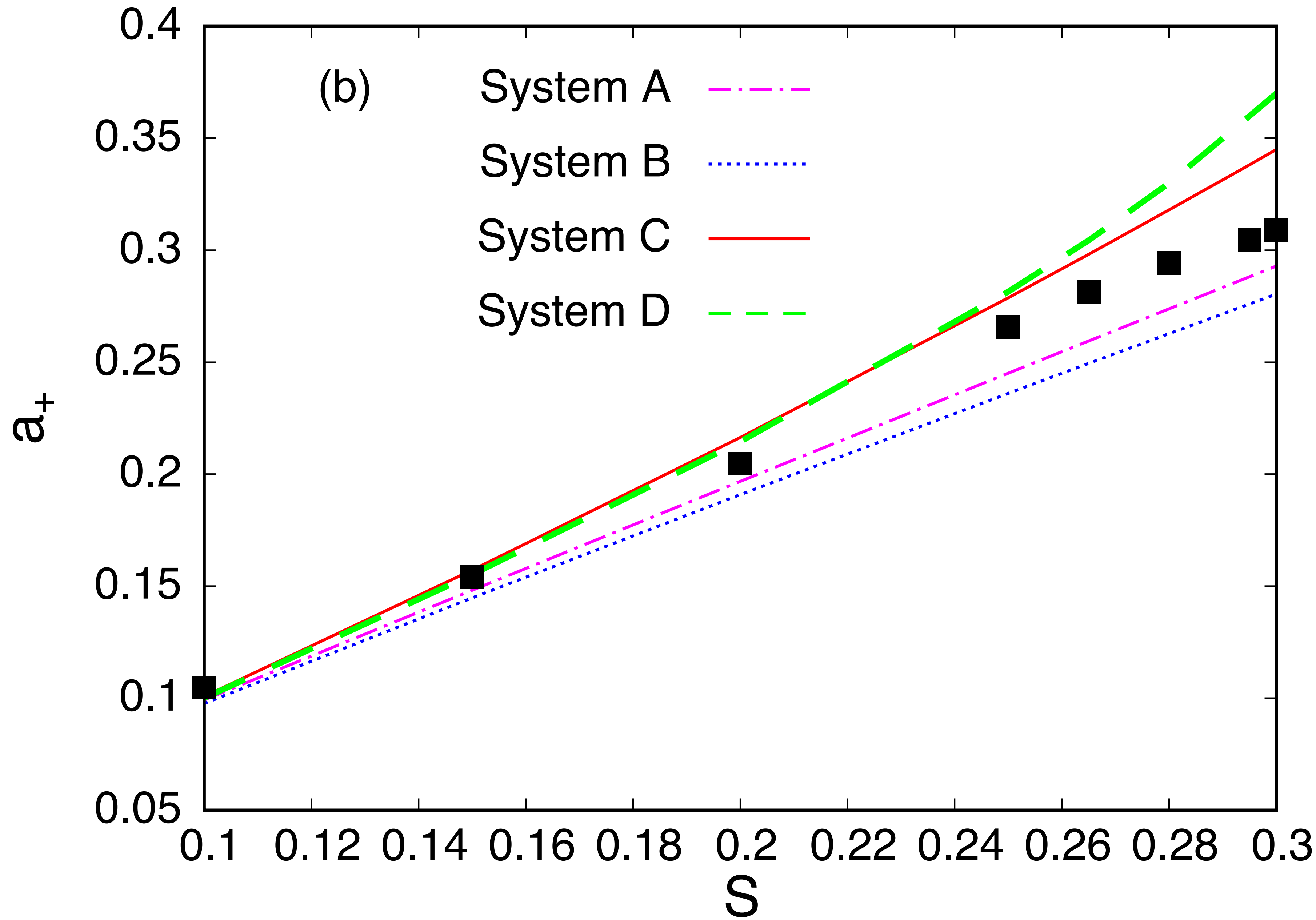

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